Equilibria in Models of Binary Choice with Heterogeneous Agents and Social Influence

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ABSTRACT. *In this paper we explore the impact of social influence on the equilibria in the simplest model with binary choices. We investigate the interplay between social influence and individual decision to adopt with global interactions, both analytically in the case of an infinite number and numerically in the case of a finite number of agents. We derive exactly necessary and sufficient conditions for the existence of single or multiple equilibria as a function of the strength of social influence. In particular, this model encompasses the classical downward sloping demand curve as a particular case. The paper also provides a brief introduction to avalanches and hysteresis loops, and includes extensive bibliographical references.*

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1. Introduction

The question of how social influence (externalities) affects the individual choice, or, in other words, the trade-off between "Individual Strategy and Social Structure" (Young, 1998) is now on the economists' agenda (Arthur *et al*., 1997; Durlauf and Young, 2001; Bourgine and Nadal, 2004). The simplest form of this question, the existence of a critical mass in binary choices with externalities, has been first addressed in the seminal (Schelling, 1973, 1978). The question of individual and collective thresholds of adoption was introduced later by (Granovetter, 1978) and (Granovetter and Song, 1983, 1986).

In the example of a riot, used by (Granovetter, 1978), agents have to choose between participating or not to a riot. Each agent has an *individual threshold of adoption* (a subjective cost), which corresponds to the number of people in the riot he considers to be sufficient to modify his behaviour (i.e. to join the riot). In his simple example, Granovetter assumes a population of *100* people with a uniform (discrete) distribution of the thresholds (an agent with threshold *0*, an agent with threshold *1*, an agent with threshold *2*, and so on, the last agent having threshold *99*). The individual with threshold *0* is the instigator of the dynamics. He decides to adopt a deviant behaviour, i.e. for example, to break a window. As a consequence, the agent with threshold 1 joins him, which in turn leads the agent with threshold 2 to modify his behaviour, and so on. Gradually, the riot grows and reaches the equilibrium where all the population is affected. This mechanism is often called a "domino effect" or a "bandwagon effect" (Leibenstein, 1950; Rohlfs, 2001). In physics literature, one speaks of an "avalanche".

In the example above, the distribution is uniform and the population is regularly distributed across discrete thresholds in the sense that there is one agent for each integer between 0 and 99. Then, the equilibrium avalanche size is equal to the total population. If the distribution of the thresholds is modified, the equilibrium may change. Let us suppose e.g. that the individual with threshold 2 does not exist. Then, the dynamics is limited to the first two people with the thresholds 0 and 1. In this case of a finite size and discrete distribution, the revision of the situation of only one person in the population may lead to a deep change in the global behaviour and exclude or at least reduce the size of avalanches.

As shown in the previous example, the collective behaviour of a population of interacting agents is non trivial. The aggregate outcome may be characterized by *multiple equilibria* and *complex dynamics* with "tipping" and "avalanches". As a result, in the deterministic context, the final equilibrium depends on the distribution of individual thresholds, the strength of social interactions, and in numerous cases with several equilibria, the selection of a particular equilibrium depends on the history of the collective dynamics. In order to contribute to the understanding of such a collective process, we use in this paper methodologies taken from both the *population game* and the *complex systems* approaches (Weisbuch, 1991). We introduce a simple model of binary choice with social influence, which allows us to revisit formally the Granovetter's model of riot. In this model agents have to make a binary choice, "to adopt" or "not to adopt" (concretely, the choice may be to adopt or not a new technology, to vote for or against the new constitution, to buy or not a good, etc.) under social influence. The latter may be local or global depending on the neighbourhood structure. The agents' preferences are both *intrinsically heterogeneous* (the agents have different preference structures) and *interactively heterogeneous* (their preferences depend on the choices of their neighbours).

Individual choice and social influence: precursors and surveys

In addition to Schelling and Granovetter (and Song) there are numerous other contributions to the modelling of social interactions that are related to our approach. In mathematical sociology (Weidlich and Haag, 1983) proposed a generic model of stochastic opinion formation with global social influence based upon the master equation and the Fokker-Plank approximation. (Kindermann and Snell, 1980) identified a social network as an application of a Markov random field. A very significant contribution is (Galam *et al.*, 1982), who proposed a pioneering application of statistical physics to sociology. In the context of strikes, using the framework of critical phenomenon from physics, the authors identified the existence of a *critical point* in the neighbourhood of which the system's behavior is extremely sensitive to small changes in parameters as well as to the history of the system. Around this point, small microscopic changes in the initial conditions can lead to drastic changes at the macro level due to tipping effects. Further works appeared in the sociological literature in the 90s. For a survey see: (Marwell and Olivier, 1993; Olivier and Marwell, 2001), and more recently (Sampson and Morenoff and Gannon-Rowley, 2002) among others.

In economics, the pioneering work of (Fölmer, 1974) applied local stochastic interactions via Markov random fields to a general equilibrium model with uncertainty. The same year, Gary Becker considered the impact of the social environment and of social interactions in the rational decisions of individuals through his concept of "social income" (Becker, 1974). In the middle of the 80's, (Kirman, 1983) and (Kirman and Oddou and Weber, 1986) suggested using stochastic graph theory in order to take into account the local communications between agents within markets. The real take-off for the models of individual choice with social influence in economics was in the 90's, in particular with Durlauf and co-authors (i.e. Durlauf, 1991, 1993; Blume and Durlauf 2003; Brock and Durlauf, 2001a) and (Blume, 1993, 1995). The social dimension in the Beckerian tradition has been developed more specifically by Becker himself (Becker, 1991; Becker and Murphy, 2000) and more formally by (Glaeser and Sacerdote and Scheinkman, 1996; Glaeser and Scheinkman, 2002). The question of the local topologies of interactions has been addressed by (Elisson, 1993; Ioannides, 1997, 2006) among others. Recently, some theoretical advances in various fields including random networks must be attributed to Horst and co-authors (see among others Horst and Scheinkman, 2005). For syntheses see (Blume, 1997; Durlauf, 1997, 1999; Ioanides, 1997; Kirman, 1997, 2006; Blume and Durlauf, 2001; Brock and Durlauf, 2001b; Phan *et al.,* 2004). On the empirical side there is also a growing interest for the role of social – or "non market" – interactions in individual decision-making as well as on the resulting effects on the outcomes (Manski, 2000). This literature addresses a great scope of subjects such as fish wholesale market (Nadal *et al*., 1998), labour market behaviour and unemployment patterns (Topa, 2001; Conley and Topa, 2002; Oomes, 2003; Ioannides and Loury, 2004), urban economics and housing demand (Ioannides and Zabel, 2003), smoking (Krauth, 2006a-b; Soetevent and Kooreman, 2006). For syntheses on the econometric literature, see (Manski, 1993; Brock and Durlauf, 2001b; Blume and Durlauf, 2005; Soetevent, 2006).

In the first part of the paper, we introduce our general framework for models of binary choice with externalities (section 2.1). In section 2.2 we compare our model with a class of models introduced by Durlauf and co-authors, see in particular (Brock-Durlauf, 2001a). In section 2.3 we show how our framework is related to population games. In the second part, we investigate analytically for the system of an *infinite number of agents* the equilibrium properties of the model under global influence and triangular distribution of the idiosyncratic preferences (section 3.1). In section 3.2 we use our framework to formalize the Granovetter's example of riot dynamics in the context of the myopic best reply. In the third part, we give a brief introduction to avalanches and hysteresis loops (section 4.1). In section 4.2 we present numerical simulations for *finite size populations* for a global influence network. These simulations were conducted by using the multi-agent platform "Moduleco-Madkit" (Gutknecht and Ferber, 2000; Phan, 2004; Michel *et al.*, 2005).

2. From Local to Global Behaviour

The basic model of binary choices with externality presented here (the "*GNP model"*) is based on (Gordon *et al*., 2005; Nadal *et al*., 2003; 2005; Semeshenko *et al*., 2006) and on the generalization to a large class of distributions in (Gordon *et al*., 2006). It allows to study the collective behaviour of a population of interrelated heterogeneous agents.

2.1. Modelling the individual choice in a social context

We consider a set of N agents $i \in \Lambda_N \equiv \{1,2,...N\}$ with a classical linear willingness-to-adopt function. Each agent makes a simple binary choice, either to adopt ($\omega_i = 1$) or not ($\omega_i = 0$) (e.g., to buy or not one unit of a single good on a market, to adopt or not the social behaviour, etc.). A rational agent chooses ω_i in order to maximize his surplus:

$$\max_{\omega_i \in \{0,1\}} \left[\omega_i V_i\left(\tilde{\omega}_{-i}\right) \right] \qquad with: \quad V_i\left(\tilde{\omega}_{-i}\right) = \left(H_i - C\right) + \frac{J_{ik}}{N_{\vartheta i}} \sum_{k \in \vartheta_i} \tilde{\omega}_k \qquad [1]$$

where C is the cost of adoption, assumed to be the same for all agents, and H_i is the idiosyncratic preference component. The cost of adoption can be subjective or objective - it may e.g. represent the price of one unit of a good. Each agent i is influenced by the (expected) choices $\tilde{\omega}_k$ of his neighbours $k \in \vartheta_i$, within a neighbourhood $\vartheta_i \in \Lambda_N$ of size $N_{\vartheta i}$. Denoting $J_{ik} / N_{\vartheta i}$, the corresponding weight, i.e. the marginal social influence on agent i from the decision of agent $k \in \vartheta_i$, the social influence is a weighted sum of $\tilde{\omega}_k$ choices. When the weights are assumed to be positive, $J_{ik} > 0$, it is possible, according to (Durlauf, 1997), to identify this external effect as *strategic complementarities* in the agents' choices (Bulow, Geanakoplos and Klemperer, 1985).

For simplicity, we consider here only the case of *positive homogeneous influences*: $\forall i \in \Lambda_N, \forall k \in \vartheta_i : J_{ik} = J > 0$. For a given neighbour k the social influence is $J / N_{\vartheta i}$, if the neighbour is an adopter ($\omega_k = 1$), and zero otherwise. Let $m^e_{\vartheta i}$ be i's expected adoption rate within the neighbourhood and η be the total rate of adoption in the population.

$$m^e_{\vartheta i} \equiv m^e_{\vartheta i}(\tilde{\omega}_{-i}) \equiv \frac{1}{N_{\vartheta i}} \sum_{k \in \vartheta_i} \tilde{\omega}_k \qquad \eta \equiv \eta(\omega_i = 1) \equiv \frac{1}{N} \sum_{i \in \Lambda_N} \omega_i \qquad [4]$$

With these assumptions the surplus of agent i if he adopted is: $H_i - C + J m^e_{\vartheta i}$.

2.2. Fixed vs random idiosyncratic heterogeneity

The general structure of the GNP model in [1] is reminiscent of a class of models by Durlauf and co-authors (Blume and Brock, 2001) and especially (Brock and Durlauf, 2001a, 2001b) - hereafter referred as the BD model. But this apparent similarity is only superficial, because the structures of BD and GNP models differ by the *nature of the disorder* (e.g., the heterogeneity across agents and randomness). The BD model belongs to the classes of both Random Utility Models (RUM) and Quantal Choice Analysis (QCA)[1]. The utility is additively stochastic: the

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[1]The *Random Utility Model* (RUM) finds its origin in Thurstone's model of comparative judgment (Thurstone, 1927). Thurstone's *law of comparative judgement* is a mathematical representation of the *discriminal process.* In this process each binary choice has a specific random component, following its own random law. The axiomatic version of the RUM - the "strict utility" model of (Luce, 1959) was introduced in economics by (Marschak, 1960; Block and Marschak, 1960). The *Quantal Choice Analysis* (QCA) derives a logistic distribution for the choice function from the RUM. More specifically, (McFadden, 1974)

idiosyncratic preferences of agents are time-dependent i.i.d. variables. The individual preferences have an identical deterministic part, and the exogenous heterogeneity across agents comes from the random term of the RUM only[2].

In the GNP model agents are heterogeneous with respect to their idiosyncratic preferences, which remain fixed and do not contain additively stochastic term. The *Idiosyncratic Willingness to Adopt* (IWA) of each agent is distributed according to the Probability Density Function (pdf) $f_y(y)$ of the auxiliary centred random variable Y, such as H is the average IWA of the population:

$$H_i = H + Y_i \qquad with: \quad \lim_{N\to\infty} \frac{1}{N}\sum_N Y_i = 0 \qquad \Rightarrow \qquad \lim_{N\to\infty} \frac{1}{N}\sum_N H_i = H \qquad [5]$$

As Y_i remains fixed, the resulting distribution of agents over the network of relations is a *quenched random field*: the agents' choices are purely deterministic[3]. This contrasts with the random utility approach in the BD model, mentioned before. As underlined elsewhere (Phan *et al*., 2004; Nadal *et al*., 2003, 2005) these two approaches are quite different. In a situation where agents make their myopic choices repeatedly at each time, they may lead to different behaviours (Stanley, 1971; Galam and Aharony, 1980, 1981; Galam, 1982; Sethna *et al*., 1997, 2005). One of the advantages of the GNP model is that the qualitative feature of the results may be generalized to a large class of distributions (Gordon *et al.*, 2006)[4]. We can assume hereafter without loss of generality that the idiosyncratic preferences are distributed according to a triangular pdf. This allows the analytical determination of the equilibrium properties. The interest of studying such idiosyncratic (exogenous) heterogeneity becomes clearer if one reinterprets the GNP model within a game theoretic framework. One can identify the present GNP model with classic configurations that exist in game theory.

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establishes a necessary and sufficient condition for this derivation: if all random arguments of the underlying RUM model are i.i.d. double exponential distributed, the joint distribution is logistic, see also ((Manski, 1977; Yelott, 1977; Anderson et al., 1992). According to RUM and QCA, (Brock and Durlauf, 2001a) assume the extreme value, type I distribution (that includes the double exponential) for the random term. Hence, based on the difference of these random variables, *the choice in the BD models follows a logistic distribution function*.

[2] From the physicist's point of view the BD model belongs to the class of the Classic Ising Models at finite temperature, with "annealed" (entropic) disorder.

[3] The GNP model corresponds to the Random Field Ising Models (RFIM) at zero temperature in physics. Another socio-physics model with this property is (Galam, 1997)

[4] According to RUM and QCA (see note 1), in the BD model the logistic distribution concerns the *random choice*, while in GNP the IWA is the exogenous *deterministic part* of the surplus function. It would be possible to generalize both GNP and DB model by adding to the heterogeneous IWA of the GNP (with a large choice of possible distribution) a McFadden-RUM-like (extreme value, type I, hence logistic by addition) random component.

2.3. Individual interactions in the neighbourhood as a population game

Agents use a best response function given their expectations about the behaviour of their neighbours: $\tilde{\omega}_{-i}$. Each agent i has only *two possible strategies*: to adopt $\omega_i = 1$ or not $\omega_i = 0$. The best response function of an agent playing against "the field" is formally equivalent to that of an agent playing against a *Neighbourhood Representative Player* (*NR*). NR player in turn plays a *mixed strategy* $\omega_{nr} \in [0,1]$[5] given by the proportion of players in the neighbourhood choosing the strategy $\omega_i = 1$. Let $m^e_{\vartheta i}(\tilde{\omega}_{-i})$ be the expected proportion of i's neighbours who play the pure strategy "adopt". Thus, player i plays against the *mixed strategy*: {adopt with probability $m^e_{\vartheta i}(\tilde{\omega}_{-i})$, not adopt with probability $1 - m^e_{\vartheta i}(\tilde{\omega}_{-i})$ }. Then we can construct the "normal form" payoff matrix G1 of a player i against this NR player. According to (Blume, 1997), this later interpretation of the mixed strategy must be related to the framework of population games. In Table 1, the matrix G1 indicates the total (cumulated) payoff of the agent i as a function of his own strategy choice and of the strategy used by the *fictitious* player *NR*. According to our interpretation a cooperating (respectively defecting) NR is equivalent to cooperation by all neighbours of agent i leading to a total payoff for i of $H_i - C + J$ (respectively $H_i - C$). In all cases, let us remark that if $\omega_i = 0$ the choices of the neighbours do not have any effect on agent i's payoff.

A game in which each player has the same number of strategies and a common interest in coordinating on the same strategy is called a "coordination game". In such a game the number of pure Nash equilibria is equal to the number of available strategies. For example, the game G1 of the table 1.a is a coordination game if all i players have both $H_i - C < 0$ and $H_i - C + J > 0$. In that case there are two Nash equilibria: everybody adopts ($\omega_i = 1$ for all i), and nobody adopts ($\omega_i = 0$ for all i). The cost of a unilateral deviation from the coordination equilibrium is $H_i - C + J$, if everybody adopts and $C - H_i$, if nobody adopts. In the case of a heterogeneous population with bounded support of IWAs ($Y_i \in [Y_{min}, Y_{max}]$), it is interesting to identify the values of the parameters $H - C$ and J for which we have a coordination game. To that purpose, it is convenient to normalize the payoff function so that the off diagonal payoff corresponding to the uncoordinated strategies is zero. The resulting modified matrix game is called a "pure coordination game" if the diagonal payoffs that correspond to the coordinated strategies are positive.

More formally, according to (Monderer and Shapley, 1996), the best-reply sets and dominance-orderings of the game G1 are unaffected if a constant term is added to a column (i.e. $C - H_i$), and if all the columns are multiplied by a constant (i.e. $N_{\vartheta i}$). The matrix G2 in Table 1.b is said to be "best reply equivalent" to the matrix G1 of Table 1.a**.** The distribution of equilibria is the same in both cases. If the first is a coordination game, the second is one also. However, the diagonal of G2 does not indicate the cumulated payoffs, contrary to the diagonal of G1. But, as pointed out in

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[5] In the case of finite neighbourhood, ω_{nr} takes its value in a discrete subset of [0, 1]. For example if N = 2, we have $\omega_{nr} \in$ {0, 1/2 ,1}

the previous paragraph, the diagonal values are a direct measure of the cost (the risk in the sense of Harsanyi and Selten, 1988) of a unilateral deviation from the coordination solution ($\omega_i = \omega_{nr}$) in the case of the pure strategy framework.

Table 1. *Payoff matrix for an agent i and best reply equivalent potential game*

a - game G1	$\omega_{nr} = 0$	$\omega_{nr} = 1$
$\omega_i = 0$	0	0
$\omega_i = 1$	$H_i - C$	$H_i - C + J$

b - game G2	$\omega_{nr} = 0$	$\omega_{nr} = 1$
$\omega_i = 0$	$C - H_i$	0
$\omega_i = 1$	0	$H_i - C + J$

Player *i* in rows, fictitious NR Player - indexed *nr* - in columns

In the present case of bounded distribution, the existence of the lower bound (Y_{min},) and the upper bound (Y_{max}) for the distribution of the centred random variable Y, may allow an identical behaviour in the population in spite of heterogeneous IWAs. By dominance analysis we can identify the classes of agents with the same behaviour, for a given cost C and a particular value of the IWA. If $C - H > Y_i + J$, then *never adopt* ($\omega_i = 0$) is the strictly dominant strategy. Agents who never adopt (for any $m^e_{\vartheta i}$) are said to be of type (0). If $H - C > -Y_i$ *always adopt* ($\omega_i = 1$) is the strictly dominant strategy. Agents who always adopt (for any $m^e_{\vartheta i}$) are said to be of type (1). If $C > H_i > C - J$ then the surplus of adoption $H_i - C + Jm^e_{\vartheta i}\left(\tilde{\omega}_{-i}\right)$ may be either positive or negative depending on the values of the expected rate of adoption within the neighbourhood $m^e_{\vartheta i}\left(\tilde{\omega}_{-i}\right)$. These agents are *conditional adopters* and said to be of type (2). The relevant cases are the ones with (at least some) agents of type (2).

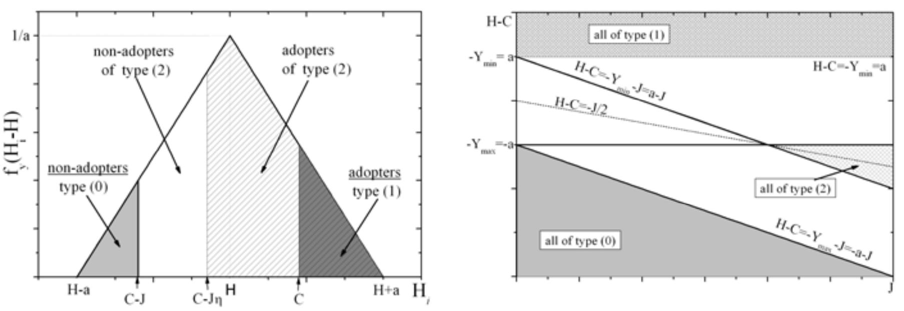

Figure 1. *Distribution of agents with respect to their type, on the pdf and in the space (J, H – C) for the symmetric triangular distribution on the interval [-a, a].*

Figure 1.a presents a (symmetric triangular) distribution of all the 3 types for a given cost C. The agents of type (0) never adopt (light grey on the left) and agents of type (1) always adopt (dark grey on the right). Within the agents of type (2) only those with $H_i - C + Jm^e_{\vartheta i} > 0$ will adopt, i.e. these are the agents that adopt thanks to social influence (hashed region).

Let us focus now on the Nash equilibrium properties in the case of *global influence*, where $m^e_{\vartheta i}\left(\tilde{\omega}_{-i}\right) = \eta$ for all *i*. Figure 1.b exhibits a distribution of agents' type in the space (J, $H - C$) for the symmetric triangular distribution[6]. In the south-west light grey zone there are only agents of type (0), while in the north-dark-zone there are only agents of type (1). Accordingly, even if the population is heterogeneous, in this mono-type configuration, the Nash equilibrium is either full adoption or non-adoption. In the white zone there is a mixture of at least 2 types of agents, with necessarily some agents of type (2). In the north-east zone, where $H - C > -Y_{min} - J$ (condition 0), there is no agent of type (0), and the surplus is positive for all agents with $\eta = 1$. Full adoption is consequently Nash equilibrium in that zone. Conversely in the south zone, where $H - C < -Y_{max}$ (condition 1), there is no agent of type (1). This latter condition corresponds to a negative surplus for all agents if nobody adopts ($\eta = 0$). Non-adoption is consequently Nash equilibrium in that zone.

If both conditions hold then *all agents* are of type (2), and they all have the same structure of best response. Despite the heterogeneity among agents, we have typically a *coordination game* with two Nash equilibria ($\eta \in \{0, 1\}$). This is the case with $-Y_{min} - J \leq H - C \leq -Y_{max}$, corresponding to the hashed triangular zone in the east on figure 1.b. This implies a sufficiently strength intensity of social effect, with respect to the dispersion of the preferences, that needs to be relatively moderate: $J > Y_{max} - Y_{min}$. Finally all the 3 types of agents coexist in the west zone, where $-Y_{min} - J > H - C > -Y_{max}$, (like on figure 1.a). If a Nash equilibrium exists, some agents will be adopters and some others non–adopters (due to the irreducible existence of agents of both type (0) and type (1)). Since different strategies must coexist at the equilibrium (if the latter exists and is stable) we say that it is a "*hybrid*" Nash equilibrium (see examples in the following sections). More knowledge about the existence and possible multiplicity of equilibria involve the analytic considerations, hereafter presented in section 3.

In order to discuss further coordination issues, one may consider the special case where all agents have the same IWA: $H_i = H$ for all *i.* The game is now symmetric and the matrix G2 is particularly useful. The equilibrium with the higher relative cost is said to be "risk dominant". In a deterministic selection process with myopic best reply, the higher the relative value of this cost, the larger the corresponding attractor. For $C - H = J/2$, we have the *risk neutral situation* (each attractor has the same size, see i.e. section 3.2 fig 5.d). For $C - H > J/2$, the *Pareto dominant* Nash equilibrium (everybody adopts) is said to be *risk dominated*, and the corresponding attractor is smaller than the other one[7] (nobody adopts - see i.e. section 3.2 fig 5.f). This latter game is well known in game theory as the "Stag Hunt game". All this

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[6] The following properties only stand for a bounded support that allows the possibility to have one or two types of agents excluded for a particular value of the cost C. In the case of unbounded support (i.e. logistic, Gaussian, or any other) there is always a residual proportion of each type for finite values of C.

[7] In a stochastic context with "enough" noise, the Pareto dominated / risk-dominant equilibrium (nobody adopts) is the only stochastically-stable one (Blume 1997, Young 1998).

considerations remain valid in the case of the agents of type (2). The stag hunt zone is within hashed triangular grey zone below the dashed line, for $-J/2 \geq H - C \geq a - J$. In the present case of bounded distribution, the dominance-ordering analysis allows us to predict the issues of some classic configurations (i.e. symmetric Nash equilibrium), where all agents have the same structure of best reply. But this may be done for only some special cases. In other more general situations, the outcome of the collective process is not easily predictable. A more general analysis requires the use of a tool from statistical mechanics (see in the section 3).

3. Collective Behaviour (I): the global externality with large population

At the collective level we suppose that agents have to make their binary choices repeatedly at successive time steps t following the usual parallel (synchronous) updating. In the global externality case (full connectivity) each agent *i* bases the expectation of the fraction of adopters among other agents $m^e_{\vartheta i}\left(\tilde{\omega}_{-i}\right)$ on the average behaviour of all other individuals: $m^e_{i,k} \equiv \sum_{k \neq i} \tilde{\omega}_k / \left(N-1\right)$. In the large N limit we may approximate the effective (observed) $m^e_{\vartheta i}$ with the total fraction of adopters (called *mean-field* approximation in statistical mechanics), and therefore: $m^e_{i,k} \approx m^e_i \equiv N^e_i / N,$ where N^e_i is the agent's expected number of adopters in the population. Thus, it is convenient to assume that current agents' expectations follow some similar kind of mean-field approximation. For the purpose of the following dynamic considerations, we therefore restrict to the simple case of *myopic expectations*, i.e. $m^e_i(t) = \eta(t-1)$ for all agents, where $\eta(t-1)$ is the true observed fraction of adopters at the previous time step. In this dynamic setting the individual surplus can be simplified as:

$$V_i\left(\eta(t-1)\right) = H_i - C + J\eta(t-1) \qquad [7]$$

3.1. Static analysis: possible multiplicity of equilibria

The static analysis focuses on the equilibrium state $\eta^* : \eta(t) = \eta^*, \forall t \geq t^*$. It is convenient to identify a *marginal adopter*, which is indifferent between adopting and not adopting. Let $H_m = H + Y_m$ be its IWA; this marginal adopter has zero surplus $W_m = \omega_m V_m = 0, \forall \omega_m \in \Omega$ such that:

$$Y_m = C - H - J\eta^* \qquad [8]$$

Consequently, an agent i adopts if $Y_i > Y_m$ and does not otherwise.

The rate of adoption is given by the solutions of the self-consistent equation:

$$\eta^* = P\left(Y_i > Y_m\right) \equiv G_y(Y_m) \equiv \int_{max\{Y_{min},Y_m(\eta^*)\}}^{max\{Y_{max},Y_m(\eta^*)\}} f_y\left(y\right)dy \qquad [9]$$

Where $\eta^* = 1$ if $Y_{min} > Y_m$ and $\eta^* = 0$ if $Y_{max} < Y_m$.

The fixed point condition [9] has one or three solutions, with two stable equilibria in the latter case. In the following we only adduce the main issues (Appendix A-, B provide the detailed calculus for both symmetric (A) and asymmetric (B) distributions of IWA). Denoting $G_y^{-1}(.)$ the inverse of $G_y(.)$, we have the following equivalence :

$$\eta^* = G_y(Y_m) \Leftrightarrow Y_m = G_y^{-1}(\eta^*) \qquad [10]$$

Condition [8] (zero surplus for the marginal adopter) can be rewritten as:

$$H - C = D\left(\eta^*,J\right) \equiv -G_y^{-1}(\eta^*) - J\eta^* \qquad [11]$$

Figure 2 illustrates the parametric graphs of the function $D(\eta,J)$ for different values of the social influence weight J, for the symmetric triangular distribution in the interior of $[-a,+a]$ (*with* $-Y_{min} = Y_{max} = a = 2$). There are equal proportions of individuals willing to adopt or not the high values of $H - C$. Solutions to [9] correspond to the intersections of the parametric graph $D(\eta,J)$ with the horizontal line $H - C$. A sufficiently strength of the social effect, given by J, is a necessary condition for multiplicity (non monotonic graphs of $D(\eta,J)$). But in addition, the average surplus without externality $H - C$ must be sufficiently negative to have to be compensated by the positive social effect, for high level of adoption. Conversely, as pointed out by (Glaeser and Scheinkman, 2002) a *moderate social influence* is a sufficient condition of uniqueness (monotonic graphs of $D(\eta,J)$).

In Levy (2005) a necessary condition of multiplicity (e.g. phase transition) is the existence of η such as $dG_y(Y_m)/d\eta \geq 1$. In the present case of a symmetric triangular distribution of IWA this implies $Jf_y \geq 1$, or equivalently: $J \geq 1/f_y$. Given the maximum of the distribution function: $f_y^{max}(y) = 1/a$, the necessary condition becomes: $J \geq a = J_B$.

For $J < J_B$ the function $D(\eta,J)$ increases monotonically with η. The fraction of adopters is uniquely defined for each value of the weight J. For large values of $J > J_B$, the line $H - C$ intersects three times the representative curve of $D(\eta,J)$ (in the dark grey cross-hatched zone). There are two stable solutions, either a low $0 \leq \eta^- < 0.5$ or a large fraction of adopters $0.5 < \eta^+ < 1$, and an intermediate unstable solution $\eta = 0.5$. These stable solutions are *hybrid* Nash equilibria, where two different strategies may coexist. For values of $J > J_c \equiv 2a$ a *hybrid* Nash equilibrium may coexist for intermediate (negatives) values of $H - C$ either with the full adoption ($\eta = 1$)

or no adoption at all ($\eta = 0$) (the light grey horizontal-hatched zone). We have also the possibility to have a pair of coordination equilibria with full adoption / no adoption, in the case identified in section 2.3 ($-a \geq H - C \geq a - J$). For example, in the case of $J = 6$, we can observe on Figure 2 the coexistence of a very low adoption rate η^-, with a full adoption ($\eta = 1$) for values of $H - C$ not too negative ($-a/6(1 - a/6) > H - C > -a$). For lower values of $H - C$ ($-a > H - C > a - J$), we have the typical coordination equilibria identified in section 2.3, while for more negative values of $H - C < a - J - a^2/2J$, the high equilibrium η^+ disappears, and the only equilibrium is with non adoption for all agents.

To summarize the generic results, if social influence is “moderate” ($J < J_B$), there is a single equilibrium. Conversely, for sufficiently strength social effect ($J > J_B$) multiplicity *may* appear for intermediate (negative) values of $H - C$. In the cases of bounded distributions with compact support, a pair of stable equilibria: saturation to ($\eta=1$) and/or no adoption ($\eta=0$) at the boundaries of the compact support may appear.

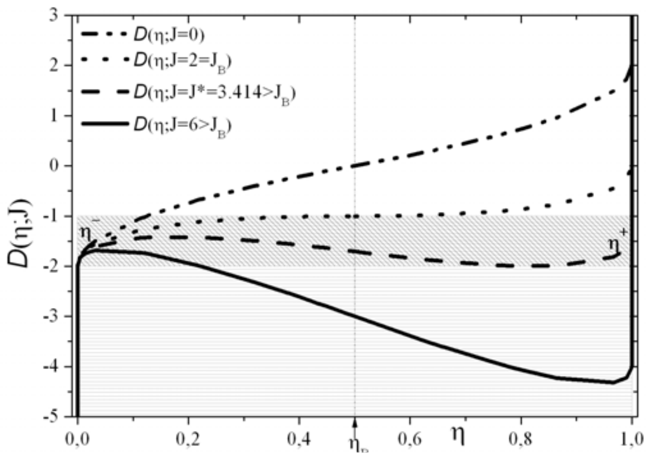


Figure 2. *Fixed points in the space (η, $H - C$) for the symmetric triangular distribution on the interval [-2, 2].*

Previous results can be graphically summarized on a phase diagram in the plane (J, $H - C$), where different regions with different regimes of Nash equilibria exist, see Figure 3 for a symmetric pdf of IWA. In accordance with partial results of section 2.3 for high values of $H - C > a - J$ (possibly low cost or/and sufficient enough strength of social influence, in the north-east region), everybody adopts and $\eta = 1$. Conversely, in the south-west zone, for low values of $H - C < -a$ (possibly high cost) and weak social influence, nobody adopts and $\eta = 0$. Analytical examination allows us to identify a new region with two Nash equilibria between $D_{min}(\eta, J)$ and $D_{max}(\eta, J)$ (including inside the coordination region that is always identified). In the eastern region with $-a < H - C < a - J$ and $J < J_B$, for a sufficiently “moderate” social effect, there is a single *hybrid Nash equilibrium*, with both non-adopters and adopters in proportion $0 < \eta < 1$. For $J > J_B$ there is a multiple solutions region delimited by the two frontiers $D_{min}(\eta, J)$ and $D_{max}(\eta, J)$ that merge at a singular point with J_B. In this region if $D_{max}(\eta, J) > H - C > D_{min}(\eta, J)$ there are two stable Nash equilibria: one with a rate of adoption lower than 50% (possibly 0%) and another one higher than 50% (possibly 100%). Accordingly, the single *hybrid* Nash equilibrium zone has two extensions for $J_B < J < J^*$, one with: $0 < \eta^- < 0.5$ if $D_{max}(\eta, J) > H - C > -a$ and another one $0.5 < \eta^+ < 1$ if $a - J > H - C > D_{min}(\eta, J)$ respectively.

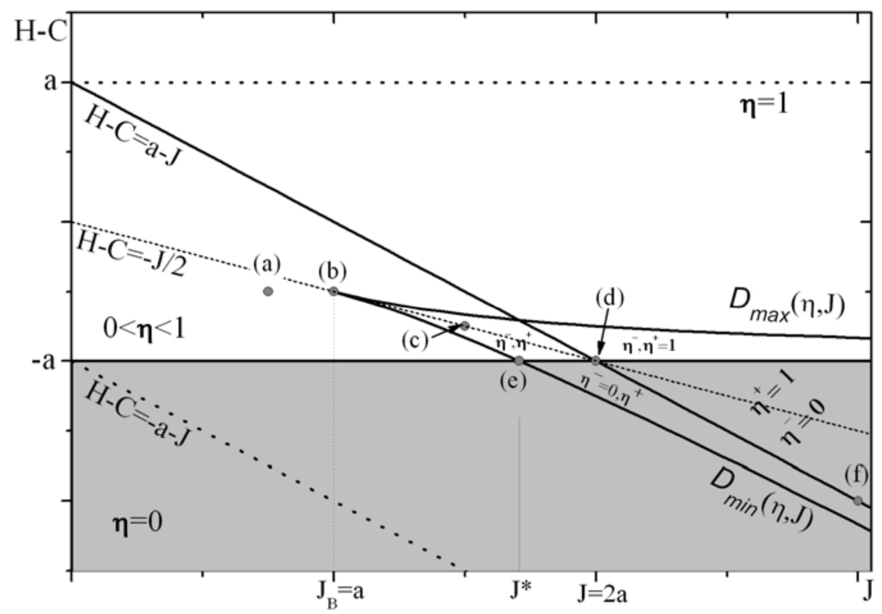
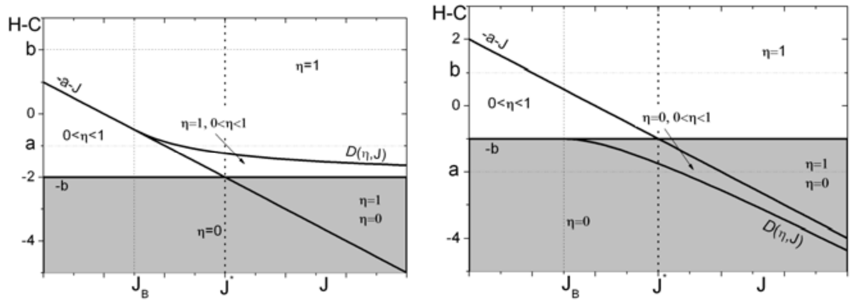


Figure 3. *Equilibrium regimes for symmetric triangular distribution in the interval [-2, 2] in the plane (J, H − C). Points (a) to (f) define the localization of the dynamic setting for Figure 6.*

The distribution of the IWA has no necessity to be symmetric. Previous results are robust even in the case of fully asymmetric pdfs. Figure 4 illustrates the phase diagrams for two limit cases of the triangular distribution of IWA, when the maximum is located at one of the boundaries. There may be few individuals willing to adopt (those having large IWA) large values of $H - C$ (Figure 4.a), or on the contrary, a majority of the population willing to adopt the high $H - C$ (Figure 4.b).

We do not detail here its calculation that follows the same lines as for the symmetric case (see Appendix B for the details). Generally we may conclude, in agreement with the above case, that for any value of $H - C > -a - J$ there is a unique solution of 100% adopters ($\eta = 1$). On the other hand, for any value of

$H - C < -b$ the solution is $\eta = 0$, nobody adopts. For a sufficiently strength of social influence $J > J^* = (b + a) > J_B$ anywhere between $-a - J \le H - C \le -b$ there is a coexistence of large fractions of adopters ($\eta = 1$) and/or none ($\eta = 0$).



(a) - pdf on [–1, 2] with a maximum at y= –1 (b) - pdf on [–2,1] with a maximum at y = 1

Figure 4. *Equilibrium regimes in the plane (J, H – C) for the asymmetric triangular pdf.*

In contrast to the symmetric case mentioned before, if the maximum is at $y = a$ (respectively, $y = b$) the boundary of the coexistence region merges with the upper $H - C = -a - J$ (respectively, the lower $H - C = -b$) frontier. In turn it changes qualitatively the regions where different equilibria exist. Hence, depending on the location of the maximum either η^+ *or* η^- solution disappears for certain range of J. In Figure 4.a for $D(\eta, J) \ge H - C \ge -b$ there is a coexistence of two solutions: either $\eta = 1$ or a finite rate of adoption $\eta < 1$. In Figure 4.b for $D(\eta, J) \le H - C \le -a - J$ the system behaviour changes from no adoption solution ($\eta = 0$) to a region where together $\eta = 0$ and a finite adoption $0 < \eta < 1$ coexist. In the coexistence region the high values of η, i.e., large fractions of adopters, correspond either to a fraction $\eta < 1$ (if $D(\eta, J) < -a - J$) or to saturation $\eta = 1$ (if $D(\eta, J) > -a - J$).

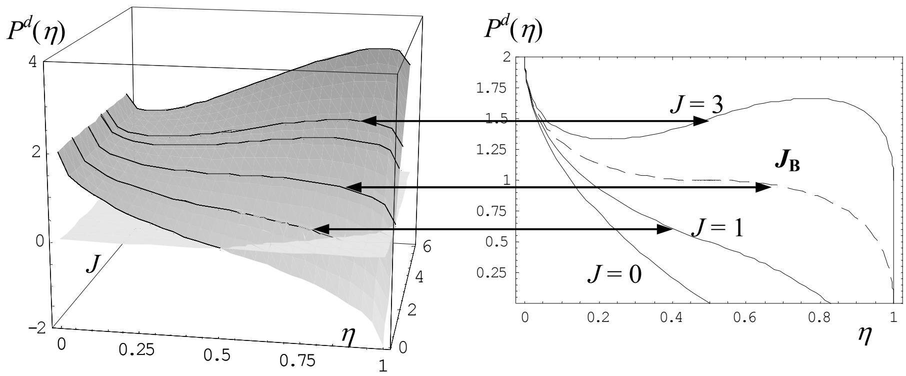
This generic model of discrete choice with social influence may be applied to the market case (early examples are Granovetter and Soong, 1986; and Becker 1991). Let C be the market price (hence $P = C$). It is possible to rewrite equation [10] in order to interpret this fixed point relation as a market equilibrium condition between supply and demand, with exogenous prices[8], written on the inverse form $P^o = P^d$. Let us define the inverse demand function:

$$P^d(\eta) \equiv H - D(\eta, J) = H + J\eta + G_y^{-1}(\eta) \qquad [12]$$

8 For the case of endogenous prices fixed by a monopolist (static optimisation), see (Nadal et al., 2005) for the logistic distribution, (Gordon et al., 2005) for the uniform distribution, and (Gordon et al., 2006) for generic distributions.

Therefore, condition [11] becomes simply:

$$P^o = P^d(\eta^*) \qquad [13]$$



(a) Function: $-D(\eta, J)$ for $0 \le J \le 5$ (b) $P^d(\eta) \equiv H - D(\eta, J)$ for $J \in \{0,1\ 2,3\}$

Figure 5. *Relationship with the "classical" downward-slopping demand curve*

Figure 5.a presents the function $-D(\eta,J)$, while Figure 5.b presents a set of parametric inverse demand functions in the case of $H = 0$ (corresponding to the upper part of the graph of $-D(\eta,J)$, over the plane $H = 0$). Figure 5.b is reminiscent to Figure 3 in the article by (Becker, 1974) on "restaurant pricing and other examples of social influences on price". However, Becker does not explicit how such a curve can be constructed, while the GNP model provides conditions under which such construction is possible. For $J < J_B$, the strength of the social influence is weak and we have a "classical" downward-slopping demand curve. Over $J = J_B$ (dashed line) the social influence is sufficiently strong to produce an upward-slopping segment, and for a given price, we have three fixed points, with only two stable equilibria.

3.2. First investigation in collective dynamics: Granovetter's riot example revisited

It is very interesting to revisit the riot model of (Granovetter, 1978) presented in the introduction within the framework of the GNP model. Let us now consider the case of a continuous distribution from equation [9]. According to [7], [8] and [9] at time t, an agent will choose to participate to the riot if the expected surplus H_i is superior to the cost of participating. Such cost is assumed to be diminished with the number of participants observed in $t-1$:

$$H_i > \left(C - J\eta(t-1)\right) \qquad [14]$$

Let us remark that an "agitator" which will take part to the riot although it appends, is the one with $H_i > C$. Conversely, an agent with $H_i < C - J$ never takes part to a riot. Then, formula [14] may be rewritten in order to make the individual threshold of the agent *i* explicit:

$$\eta(t-1) > \eta_i \equiv \frac{(C - H_i)}{J} \qquad [15]$$

Then, according to [8] the individual threshold of the marginal agent may be defined in a dynamic way as:

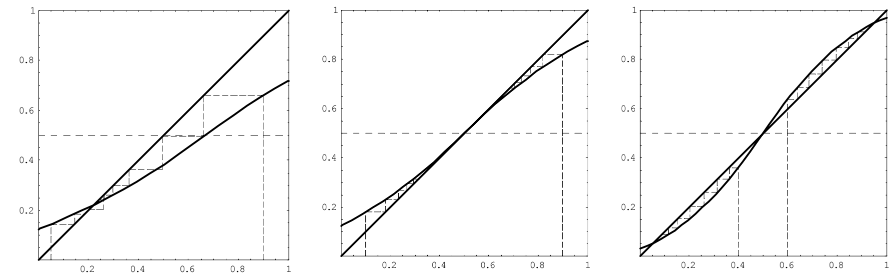
$$\eta_m(t) \equiv \frac{(C - Y_m(t) - H)}{J} = \eta(t-1) \qquad [16]$$

For a given value of $\eta\ (t\text{-}1)$, the agents which choose to participate in the riot will be those with $Y_i > Y_m(t) \equiv C - H - J\eta(t\text{-}1)$. Consequently, the rate of rioters may be defined by the way of a recurrent relationship:

$$\eta(t) = P\left(Y_i > Y_m(t) \middle| \eta(t-1)\right) \equiv G_y(Y_m(t)) \int\limits_{C-H-J\eta(t-1)}^{\infty} f_y(y)\,dy \qquad [17]$$

The recurrent relation [17] can be drawn as a fixed point dynamic graph (see Figure 6). In Figure 6.a the stable equilibrium is unique, corresponding to the case presented by Granovetter, while there are two stable *hybrid* Nash equilibria separated by an instable fixed point on Figure 6.c.

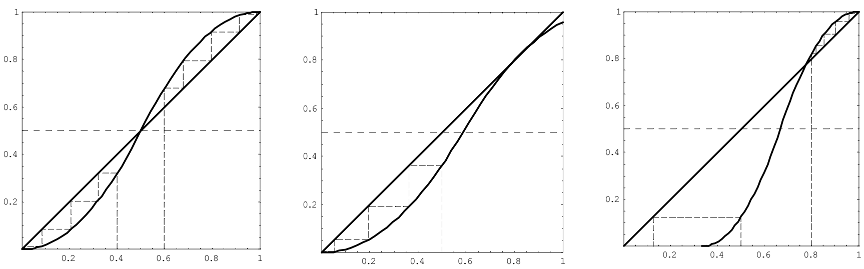
This GNP model describes the properties of many different systems (physical as well as social). It has been studied for various network architectures. It is also very interesting for its non equilibrium properties, like hysteresis loops as described below.



(a) - one stable equilibrium
$J = 1.5,\ H - C = -1$
$\eta = 1/4.5 \approx 0.22$

(b) - one stable equilibrium
$J = a = 2,\ H - C = -a/2$
$\eta = 0.5$

(c) - two stable equilibria
$J = 3,\ H - C = -1.5$
$\eta^- = 0.056,\ \eta^+ = 0.944$



(d) -- two stable equilibria $J = 4; \quad H - C = -a$ $\eta^{-} = 0; \quad \eta^{+} = 1$

(e) - one stable equilibrium $J = J^{*} = a(1 - 2^{-\frac{1}{2}})$ $H - C = -a; \eta = 0$

(f) - two stable equilibria : $J = 6, H - C = -4$ $\eta^{-} = 0, \eta^{+} = 1$

Figure 6: *Sequential dynamics of adoption for equation [17] with the symmetric triangular distribution of IWA on [-2, 2];* $\eta(t\text{-}1)$ *on abscissa and* $\eta(t)$ *on ordinate.*

4. Collective Behaviour (II): avalanches and hysteresis loops in aggregate behaviour.

In this class of choice models with externality, the adoption by a single agent in the population (a "direct adopter") may lead to a significant change in the whole population through a chain of reactions of "indirect adopters" (Phan and Pajot, 2006). The term "avalanche" is associated with such a chain of reactions, i.e. the adoption is directly induced by the behaviour change of one or several other agents and not only (and directly) by an initial variation in the exogenous conditions (i.e. costs). The corresponding mechanism of social influence has numerous examples in telecommunications and high-technology industries (see Rohlfs, 2001 for a bibliography). As a result the aggregate demand dynamics presents singular behaviour at the collective level, according to those observed in the *Random Field Ising Model* in statistical physics.

4.1. Introduction to the phase transition and hysteresis loops.

As we have seen previously, in the presence of externality and depending on the parameters, two different stable Nash equilibria ("phases" for a physicist) may exist: one with a small fraction of adopters (at the limit $\eta = 0$) and another with a large fraction (at the limit $\eta = 1$). By varying the costs a *transition* may be observed between these phases. The jump in the number of adopters occurs at different cost values according to whether the cost increases or decreases, leading to *hysteresis loops*, as presented below. In the simple case where the IWA is the same for all agents, ($H_i = H,$ for all i), the model, a *coordination game* for an economist, is equivalent to the *classic Ising model with a "uniform" external field*: $H - C$. In such a case, one would observe a "first order transition", with all the population abruptly adopting as $H - C$ passes through zero from below (and vice versa). In Figure 7, the initial threshold (increasing adoption for decreasing costs) is $C_{min} = H$, where the whole population abruptly adopts. After adoption, the (decreasing) cost threshold is $C_{max} = H + J$, where the whole population abruptly chooses $\omega_i = 0$. When all agents are adopters, cost variations between $C = H$ and $C = H + J$ have no effect on the agent's choice.

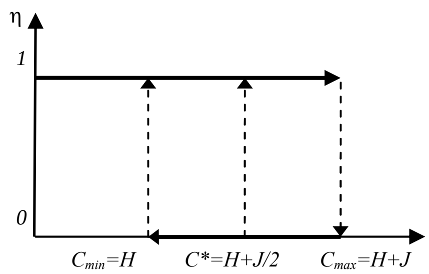


Figure 7: *Hysteresis with an identical willingness to adopt* ($H_i = H$).

From the theoretical point of view, there is a singular cost $C^* = H + J/2$ which corresponds to the unbiased situation, where the willingness to adopt is neutral on average. From a game theory point of view (section 2.3) we have a pure coordination game if C = C* (the diagonal terms of game G.2 in table 1.b are equal to J/2). As the costs of the deviation from the two coordination solutions are equal, this configuration is risk-neural. Conversely, if C > C* the coordination game is a "stag hunt". This critical point C* plays a central role in the so called *spontaneous symmetry breaking*, even when agents are only locally connected. Depending on the structure of the agent's expectations about the choices in his neighbourhood, dynamic adjustments lead all individual choices to be identical to the collective equilibrium state: either all agents adopt, or nobody adopts (Galam, 2004, Phan and Pajot, 2006).

In the presence of *quenched disorder* (fixed heterogeneous IWA: $H_i \neq H$ for all i or non-uniform external field), the number of customers evolves by a series of cluster flips, or avalanches, like in the riot described by Granovetter. If the disorder is strong enough (the variance σ^2 of the random variable Y is large compared to the strength of the social coupling J), there will be only small avalanches (each agent is sufficiently different from the others and tends to follow his own H_i). If σ^2 is small enough with respect to J, the phase transition occurs through a unique "infinite" avalanche, like in the case with identical H for all agents. In the intermediate regimes, a distribution of smaller avalanches of various sizes can be observed.

4.2 Global externality with finite size populations: ACE experiments.

We consider a simple simulation example of the discussed GNP model, using the multi-agent framework Moduleco-Madkit (Gutknecht and Ferber, 2000; Phan, 2004; Michel *et al.,* 2005). We simulate a single system (one particular realization of a random IWA) of 1156 agents with the symmetric triangular distribution of idiosyncratic preferences,, presented in section 3.1. Figure 8 shows the set of the aggregate adopters' equilibria for the system over the values of the cost, incremented in steps of 10^{-3}, within the interval [0, 4]. The simulations are done under the synchronous activation regime (all agents update their behaviour at the same time). Results (the number of adopters) correspond to the upstream (downstream) trajectories obtained upon decreasing (increasing) the cost. In this example (seed = 0, J = 4) with global influence one observes a hysteresis phenomenon with a phase transition between high and low adoption regimes around the theoretical point of symmetry breaking: $C^* = H + J/2 = 2$. On Figure 8a shows triangular (8b uniform) distributions for J = 4 and H = 0. Along the upstream equilibrium trajectory (black plots for decreasing costs) an avalanche arises for C = 1,457 (C =1.927) by a succession of cluster flips, driving the system from an adoption rate of 11% (44%) towards the total adoption (100%). Along the downstream trajectory (grey plots, for increasing costs) an avalanche arises for C = 2,477 (C = 2.023) and the adoption rate decreases dramatically from 80% (77%) to 0% (no adoption). The intermediate points in a straight hysteresis are due to the *finite size effects* and disappear in the theoretical solution obtained in the mean-field limit (when N tends to infinity) like on Figure 9.a. the are a lot in the uniform distribution of Figure 8a, only a few (at 80% and 86% in the downstream trajectory) on the triangular distribution of Figure 8.b. The irregularity of the distribution of IWA in the *finite size* case is the explanation of *finite size effects*, like in the example of the missing agent with threshold 2 in the introductory example of Ganovetter's riot. Missing agent stops the chain of reactions, as if we had too much space between two dominoes following each other. Moreover, we have observed the persistence of such an effect with a larger population (i.e. 10.000 agents).

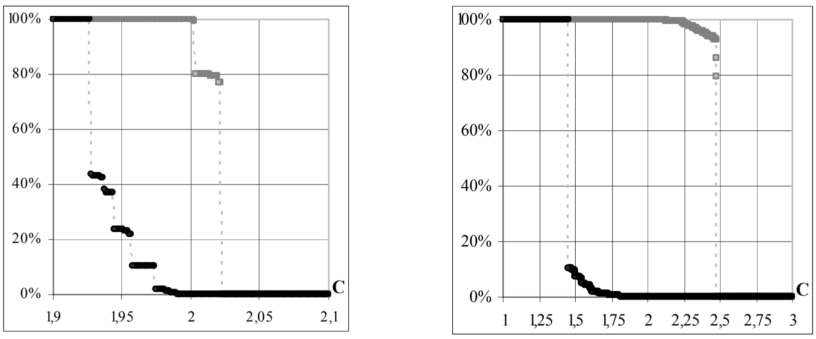
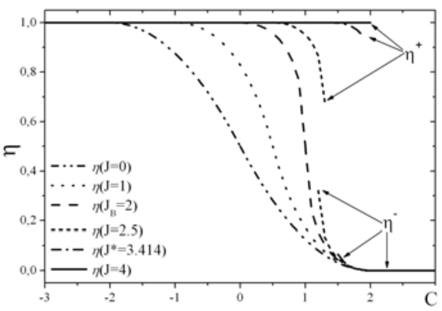


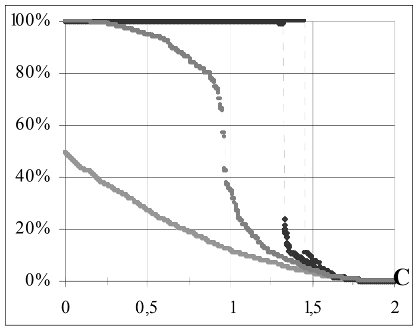
Figure 8: *Hysteresis under global externality, and synchronous activation regime (Moduleco-Madkit: 1156 agents) - Upstream (black) and downstream (grey) trajectories – Uniform (a) and Triangular(b) pseudo-random generator for J=4.*

Figure 9.a shows the theoretical mean-field equilibria for the fraction of adopters at different costs for different values of social influence J (0, 1, J_B, 2.5, J^*, 4). Figure 9.b shows a set of simulated upstream equilibrium trajectories for different values of $J > J_B$ (3, J^*, 4) in the presence of global externality. For small values of J there is no hysteresis at all. To summarize, high connectivity and strong enough social effect with respect to the dispersion of the IWA are necessary conditions for the existence of a large gap (a first order transition phase) in the distribution of the equilibrium positions. However, for finite populations (and real cases), such an effect may be attenuated by the irregular dispersion of the preferences.

There is a well established methodology to calculate the hysteresis loop starting from a saturated (ordered) state. Recently (Shukla, 2000) proposed a probabilistic method to calculate the return hysteresis exactly, starting from an arbitrary initial state. Such a result suggests new fields of investigation,as opposed to standard focus on conditions of uniqueness of equilibrium, under a *moderate social influence* assumption - see for instance (Glaeser and Sheinkman, 2002) or (Horst and Sheinkman, 2005) in the case of random systems.



(a) – theoretical equilibrium plots



(b) - upstream branch with J=0,2, J, 4*

Figure 9: *The trade-off between prices and adopters (synchronous activation).*

6. Conclusion

Most models of endogenous social interactions implicitly assume that individuals are interacting directly or indirectly with a very large group of others and thus are subject to personal or social influence. These models generally feature multiple equilibria when the marginal impact of social influence (externalities) exceeds some strictly positive critical value. Numerous papers focus on conditions of uniqueness of equilibrium, which imply a weak (or moderate) social influence. In this paper, we present also the case of strong influence with multiplicity. More precisely, while

Levy (2005) provides a necessary condition for multiplicity in a general case, we derive exactly, for some cases of a triangular distribution, necessary and sufficient conditions for existence of multiple equilibria, depending on $H - C$, J as well as the scope of idiosyncratic heterogeneity. These results are summarized on the *phase diagrams*, which represent the regions of qualitatively different collective behaviours as a function of the model parameters (J, $H - C$). Moreover, the model presents interesting equilibrium and non-equilibrium properties.

Using a well developed methodology within a framework of statistical physics, we illustrated the stationary properties for particular cases of symmetric triangular distribution of IWA under global interactions. Results of the simulation allow us to observe numerous complex dynamics on the adoption side, such as hysteresis and avalanches. This complex social phenomenon depends significantly on the structure and parameters of the relevant network. Finally, last section opens the question of *finite size effects*, also addressed by (Glaeser and Sheinkman, 2002; Krauth, 2006; Soetevent and Kooreman, 2006) among others. It would be interesting in the future to compare more systematically the analytical predictions against the simulation results and to study the statistical properties of avalanches for different values of J, like the avalanche size distribution.

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Appendix A: symmetric triangular distribution

Let us consider a triangular distribution of Y, symmetric on [–a, +a] with a maximum $f_y^{max}(y) = 1/a$ at $y = 0$.

$$f_y(y) = \frac{1}{a}\left(1 + \frac{y}{a}\right) \quad if: \quad -a < y \le 0, \; f_y(y) = \frac{1}{a}\left(1 - \frac{y}{a}\right) \; if: \; 0 < y \le a \quad \text{[A.1]}$$

and zero otherwise.

The global probability of adoption is given by the following fixed point relation (with: $0 \le \eta \le 1$)

$$\eta = P\left(Y_i > Y_m\right) \equiv G_y(Y_m) = \int_{Y_m}^{\infty} f_y(y)dy \qquad with: \quad Y_m \equiv C - H - J\eta \quad \text{[A.2]}$$

Then, with the symmetric triangular pdf, we have:

$$\eta = 1 \qquad if: \; Y_m < -a \quad \Leftrightarrow \quad H - C > a - J$$
$$\eta = 0 \qquad if: \; Y_m > +a \quad \Leftrightarrow \quad H - C < -a$$

$$\begin{aligned} \eta &= G_{y-}(Y_m) \quad if: \; -a \le Y_m \le 0 \; \Leftrightarrow \; \left(a + C - H\right)/J \ge \eta \ge \left(C - H\right)/J \\ \eta &= G_{y+}(Y_m) \quad if: \; +a \ge Y_m \ge 0 \; \Leftrightarrow \; \left(-a + C - H\right)/J \le \eta \le \left(C - H\right)/J \end{aligned} \quad \text{[A3]}$$

$$if: \; -a \le Y_m \le 0 \qquad \eta^+ = G_{y-}(Y_m) = 1 - \frac{\left(a + Y_m\right)^2}{2a^2} \qquad \text{with solution :}$$

$$\eta^+ = \frac{J(a + C - H) - a^2 + a\sqrt{a^2 - 2J(a + C - H - J)}}{J^2}, \quad \lim_{J \to 0} \eta^+ = 1 - \frac{\left(a + C - H\right)^2}{2a^2} \quad \text{[A.41]}$$

$$if:\ 0 \le Y_m \le a \qquad \eta^- = G_{y+}(Y_m) = \frac{(a - Y_m)^2}{2a^2} \qquad \text{with solution :}$$

$$\eta^- = \frac{-J(a - C + H) + a^2 + a\sqrt{a^2 - 2J(a - C + H)}}{J^2}, \qquad \lim_{J \to 0} \eta^- = \frac{(a + C - H)^2}{2a^2} \qquad [A.42]$$

By inversion of $G_y(Y_m)$ in [A.41] and [A.42] we get:

$$\begin{aligned} Y_m &= G_{y-}^{-1}(\eta) = -a\left(1 - \sqrt{2 - 2\eta}\right)\ if{:}\ Y_m \le 0,\ \eta \ge 0.5 \\ Y_m &= G_{y+}^{-1}(\eta) = a\left(1 - \sqrt{2.\eta}\right)\ if{:}\ Y_m \ge 0,\ \eta \le 0.5 \end{aligned} \qquad [A.5]$$

Using the value of Y_m following (A.2), we may define:

$$H - C = \begin{cases} D^-(\eta, J) \equiv a\left(1 - \sqrt{2 - 2.\eta}\right) - J.\eta & if{:}\ H - C \ge -J.\eta\,;\ \eta \ge 0.5 \\ D^+(\eta, J) \equiv -a\left(1 - \sqrt{2.\eta}\right) - J.\eta & if{:}\ H - C \le -J.\eta\,;\ \eta \le 0.5 \end{cases} \qquad [A.6]$$

The fixed point(s) in equation (A.2) correspond(s) to the solutions of equations (A.6). According to the configuration of function $D(\eta, J)$ there is either a single solution, or several solutions. In order to study the configurations of $D(\eta, J)$, it is convenient to study the derivatives first.

$$\frac{dD^-}{d\eta} = \frac{a}{\sqrt{2 - 2.\eta}} - J; \qquad with:\ \lim_{\eta \to 1^-} \frac{dD^-}{d\eta} = \infty \qquad and:\ \lim_{\eta \to 1/2^+} \frac{dD^-}{d\eta} = a - J$$

$$\frac{dD^+}{d\eta} = \frac{a}{\sqrt{2.\eta}} - J; with:\ \lim_{\eta \to 0^+} \frac{dD^+}{d\eta} = \infty \qquad and:\ \lim_{\eta \to 1/2^-} \frac{dD^+}{d\eta} = a - J \qquad [A.7]$$

When $D(\eta, J)$ is monotone increasing, there is a single solution. Has $D(\eta, J)$ diverged when η tends towards 0 and 1, we have three particular values on the compact interval *[0, 1]*: $\eta = 0$ *when* $H - C \le -a$; $\eta = 0.5$ *when* $H - C = -J/2$; *and* $\eta = 1$ *when* $H - C \ge a - J$.

There is also a critical value J_B beyond which several fixed points may appear. This zone begins for the smaller value of J for which function $D(\eta, J)$ has a null derivative. Then, for $J > J_B$ function $D(\eta, J)$ is no longer monotone increasing. Since the derivative of an inverse function is equal to the inverse of the initial derivative, this value corresponds to the value of η for which the pdf $f_y(y)$ is maximum, that is: $\eta_B = 0.5$, because we have: $f_y(0) = 1/a = f_y^{max}$. Then, for $\eta_B = 0.5$ we have:

$$\lim_{\eta \to 1/2^-} \frac{dD^+}{d\eta} = a - J_B = \lim_{\eta \to 1/2^+} \frac{dD^-}{d\eta} = a - J_B = \frac{1}{f_y^{max}} - J_B = 0 \qquad \text{[A.8]}$$

$$for: \quad \eta_B = 1/2 \quad \Rightarrow \quad J_B = a$$

The behaviour of $D(\eta, J)$ for different values of the social influence weighs J is presented on Figure 2 in section 3.1. For $J > J_B$, $H - C \le D(\eta, J)$ have *2* solutions in the zone of the phase space $(J, H-C)$ delimited by the two curves: $D^+_{max}(\eta_0^-(J))$ and $D^-_{min}(\eta_0^+(J))$, where $\eta_0^+(J)$ and $\eta_0^-(J)$ are the solutions of the marginal stability equation $D'(\eta, J) = 0$.

$$H - C > D^-_{min}\left(\eta_0^+(J)\right) = a - J - \frac{a^2}{2.J} \qquad 0,5 \le \eta_0^+ \le \eta^+(J) \le 1$$

$$H - C < D^+_{max}\left(\eta_0^-(J)\right) = -a + \frac{a^2}{2.J} \qquad 0 \le \eta^- \le \eta_0^-(J) \le 1/2 \qquad \text{[A.9]}$$

$$\text{with :} \quad \lim_{J \to \infty} D^-_{min}\left(\eta_0^+(J)\right) = -a \quad \text{and :} \quad \lim_{J \to \infty} D^+_{max}\left(\eta_0^-(J)\right) = a - J \qquad \text{[A.10]}$$

The representative curves of $D^+_{max}(\eta_0^-(J))$ and $D^-_{min}(\eta_0^+(J))$ cut the lines $-a$ and $a - J$ at the same point with abscissa $J^{*}=a1.707 > J_B$ (A.10 $\eta^+ = 1$) in the zone where: $-a < H - C < D^+(\eta, J)$ and two solutions ($\eta^- = 0$ and η^+) in the zone where: $a - J > H - C > D^-(\eta, J)$. In the zone where $a - J < H - C < -a$, the two solutions are: $\eta^- = 0, \eta^+ = 1$. Finally, for $H - C < D^-(\eta, J)$, $\eta = 0$ is the unique solution; and for $H - C > D^+(\eta, J)$, $\eta = 1$ is the unique solution. These properties may be summarized on the phase diagram, which defines a region with different solutions (see Figure 3, section 3).

Appendix B: asymmetric triangular distribution with a maximum at the boundaries

We consider here an asymmetric triangular pdf on the interval $y \in [a, b]$ with the unique maximum located at one of the boundaries ($f_y^{max}(y) = 2/(b-a)$ has a maximum at $y = a$, or at $y = b$).

B.1.The asymmetric pdf with the maximum at y = a.

The density distribution function is given by:

$$f_y(y) = \frac{2(y-a)}{(b-a)} \quad \textit{if: } a \le y \le b \text{ and zero otherwise} \qquad \text{[B1].}$$

The calculations and analysis follow the main lines of the symmetric case (see Appendix A). Hereafter we will introduce the main results. In the present case of a triangular distribution using the values of $G_y^{-1}(\eta) = a + (b-a)\sqrt{(1-\eta)}\ ,\ 0 < \eta < 1$, we may write a function $D(\eta, J)$ in terms of η:

$$D(\eta, J) \equiv -a - (b-a)\sqrt{(1-\eta)} - j\eta,\ 0 < \eta < 1; \qquad \text{[B.2]}$$

it is illustrated on Figure B.1.a below.

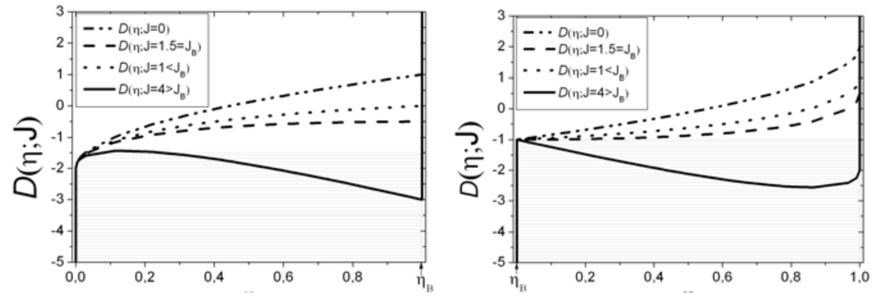

Figure B.1. *The function $D(\eta, J)$ is presented for different values of the social influence weight J. The (a) corresponds to the pdf with the maximum at y=a, the (b) corresponds to the pdf with the maximum at y=b. For the $J<J_B$ there is a single solution η for any value of H-C, while for $J>J_B$ multiple equilibria appear.*

The critical values are: $J_B = \frac{(b-a)}{2}, \qquad \eta_B = 0$. [B.3]

The demand phase diagram (see section 3 Figure 4 (a) for the pdf on [-1, 2]) is defined by

$$H - C = -a - J,\quad H - C = -b,\quad H - C = -a - J - \frac{(b-a)^2}{4J} \qquad \text{[B.4]}$$

For any value $H - C > -a - J$ a unique solution is $\eta = 1$, for $H - C < -b$ a unique solution is $\eta = 0$. In the zone where $D(\eta, J) \le H - C \le -a - J$ there are two possible solutions $\eta = 1$ and a finite $0 < \eta < 1$. Finally, in the region between $-b < H - C < -a - J$ there are either 100% of adopters $\eta = 1$, or none $\eta = 0$ (see a detailed discussion in section 3).

B.2. The asymmetric pdf with the maximum at y = b.

In the case of the asymmetric triangular distribution with the maximum at y=b the density distribution function is given by:

$$f_y(y) = \frac{2(b-y)}{(b-a)} \quad if\!: a \le y \le b \text{, and zero otherwise.} \qquad [B.5]$$

The function $D(\eta,J) = -b + (b-a)\sqrt{\eta} - J\sqrt{\eta}, 0 < \eta < 1$ is illustrated on Figure B.1.b. The critical values are:

$$J_B = \frac{(b-a)}{2}, \eta_B = 1 \,. \qquad [B.6]$$

The set of equations determining the phase diagram are

$$H - C = -a - J, H - C = -b, H - C = -b + \frac{(b-a)^2}{4J} \qquad [B.7]$$

For any values $H - C > -a - J$ a unique solution is $\eta = 1$, for $H - C < -b$ a unique solution is $\eta = 0$. In the zone where $D(\eta,J) \le H - C \le -a - J$ there are two possible solutions $\eta = 0$ and a finite $0 < \eta < 1$. Finally, in the region between $-b < H - C < -a - J$ there are either 100% of adopters $\eta = 1$, or none $\eta = 0$ (see a detailed discussion in section 3 for a pdf on [-2, 1]).