Heterogeneous Agents with Local Social Influence Networks: Path Dependence and Plurality of Equilibria in the ACE Noiseless case

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Summary. In this paper we explore the impact of local social influence on the equilibria in a model of noiseless binary choices. We investigate the interplay between social influence and individual decision to adopt. We compare probabilistic analytic calculi based upon an infinite size population and ACE based simulations for finite size populations in the case of regular local influence network. For a given exogenous value, there is a multiplicity of equilibria, depending on the previous state of the system (path-dependence). Moreover, the inner loop illustrates the return point memory effect, in which the system remembers its former state.
Keywords. Binary Choice, Complex Systems, Heterogeneous Interacting Agents, Hysteresis, Local Neighbourhood Effects, Path dependence

1 Introduction

In this paper we explore numerically by means of ACE the impact of *local social influence* on binary choices. The basic model of binary choices with externality presented here (the "*GNP model"*) is based on (Nadal *et al*., 2005; Gordon *et al*., 2005; Semeshenko *et al*., 2006, Phan and Pajot 2006; see Phan and Semeshenko 2007 for an introduction and a review of literature). GNP model has been generalized to a large class of distributions in (Gordon *et al*., 2006). It allows to study the collective behaviour of a population of interrelated heterogeneous agents. Numerous papers in this field concern homogeneous agents with stochastic choices, in particular, among others: (Brock and Durlauf, 2001) – hereafter BD model. Our GNP class of models differs by the nature of the disorder. The former belongs to the classes of Random Utility Models (RUM): the utility is stochastic. The individual preferences have an identical deterministic part and the heterogeneity across agents comes from the random term of the RUM. In our *noiseless* GNP *model*, agents are heterogeneous with respect to their idiosyncratic preferences (IWA) which remain fixed and do not contain stochastic term. This model belongs to the class of the *Quenched Random Field Ising Model*, well know in statistical physics.

The question of the local topologies of interactions has been recently examined by (Ioannides 2006). In the following, we present equilibria results for models with a local regular network (cyclical, one and two dimensional with nearest neighbourhood). This work is an extension to the case of local interactions of the GNP model presented previously. Therefore, the reader is assumed to be familiar with these references. Several important aspects of the analysis and simulation of the model which are discussed in this paper are not repeated here in order to save space, or mentioned only very briefly for the sake of completeness. Section 2 introduces the GNP model and shows how this framework is related to population games by summarizing previous contributions (Phan and Semeshenko 2007). Section 3 and 4 present and compare both probabilistic calculi for *infinite size* population and ACE based simulations for *finite size* populations in the case of simple *regular local influence network* (lattice). Calculus in section 3 are based upon a probabilistic method recently introduced by (Shukla, 2000) to calculate exactly the hysteresis path both starting from an homogeneous state (nobody adopt) or from an arbitrary initial state The simulations were conducted using the multi-agent platform “Moduleco-Madkit” (Gutknecht and Ferber, 2000; Phan, 2004). A special attention is devoted to the Sethna's inner hysteresis (Sethna *et al.,* 1993). For a given value of the external parameter (i.e. price), there is a multiplicity of equilibria, depending on the previous state of the system (*path-dependence*). Moreover, if this parameter returns back to some initial value, the system returns precisely to the same state from which it left. The inner loop illustrates the *return point memory effect*, in which the system remembers its former state.

2. GNP framework with local setting

2.1. Modelling the individual choice in a social context

We consider a set of *N* agents $i \in \Lambda_N \equiv \{1,2,...N\}$ with a classical linear willingness-to-adopt function. Each agent makes a simple binary choice, either to adopt ($\omega_i = 1$) or not ($\omega_i = 0$) (e.g., to buy or not one unit of a single good on a market, to adopt or not the social behaviour, etc.). A rational agent chooses ω_i in order to maximize its surplus:

$$\max_{\omega_i \in \{0,1\}} \left[\omega_i V_i\left(\tilde{\omega}_{-i}\right) \right] \quad with: V_i\left(\tilde{\omega}_{-i}\right) = \left(H_i - C\right) + \frac{J_{ik}}{N_{\vartheta i}} \sum_{k \in \vartheta_i} \tilde{\omega}_k \tag{1}$$

where C is the cost of adoption, assumed to be the same for all agents, and where H_i represents the idiosyncratic preference component. The cost of adoption can be subjective or objective - it may e.g. represent the price of one unit of a good. Each agent i is influenced by the (expected) choices $\tilde{\omega}_k$ of its neighbours $k \in \vartheta_i$, within a neighbourhood $\vartheta_i \in \Lambda_N$ of size $N_{\vartheta i}$. Denoting with $J_{ik} / N_{\vartheta i}$, the corresponding weight, i.e. the marginal social influence on agent i from the decision of agent $k \in \vartheta_i$, the social influence is a weighted sum of $\tilde{\omega}_k$ choices. When the weights are assumed to be positive, $J_{ik} > 0$, it is possible, according to (Brock and Durlauf, 2001), to identify this external effect as *strategic complementarities* in the agents' choices.

In the GNP model agents are heterogeneous with respect to their idiosyncratic preferences, which remain fixed and do not contain additively stochastic term. The *Idiosyncratic Willingness to Adopt* (IWA) of each agent is distributed according to $f_y(y)$ the Probability Density Function (pdf) of the auxiliary centred random variable Y, such as H is the average IWA of the population:

$$H_i = H + Y_i \quad with: \quad \lim_{N \to \infty} \frac{1}{N} \sum_N Y_i = 0 \quad \Rightarrow \quad \lim_{N \to \infty} \frac{1}{N} \sum_N H_i = H \tag{2}$$

As Y_i remains fixed, the resulting distribution of agents over the network of relations is a *quenched random field*: the agents' choices are purely deterministic. As mentioned before, this contrasts with the random utility approach in the BD model. These two approaches may lead to different behaviours (Galam, 1997; Sethna *et al.*, 1993, 2005). One advantage of the GNP model is that it does not constrain the distribution of the idiosyncratic willingness to adopt to be *a priori* logistic. Moreover, the qualitative feature of the results may be generalized to a large class of distributions (Gordon *et al.*, 2006). We can assume hereafter without loss of generality that the idiosyncratic preferences are distributed according to a bounded, triangular pdf. This allows the analytical exact determination of the equilibrium properties in the case of complete connectivity (Phan and Shemeshenko, 2007). In the following, we restrict to the case of regular nearest neighbourhood, cyclical network of dimension one (circle, with $N_{\vartheta i} = 2$) and two (torus, with $N_{\vartheta i} = 4$, von Neuman's neighbourhood). Moreover, for the sake of simplicity, we restrict to the case of *positive homogeneous influences*: $\forall i \in \Lambda_N, \forall k \in \vartheta_i : J_{ik} = J > 0$. For a given neighbour k the social influence is $J / N_{\vartheta i}$, if the neighbour is an adopter ($\omega_k = 1$), and zero otherwise. Let η_i^e be i's expected adoption rate within the neighbourhood

$$\eta_i^e \equiv \eta_i^e(\tilde{\omega}_{-i}) \equiv \frac{1}{N_{\vartheta i}} \sum_{k \in \vartheta_i} \tilde{\omega}_k \qquad (3)$$

With these assumptions the surplus of agent i if he adopted is: $H_i - C + J\eta_i^e$. The conditional probability of adoption, for a given η_i^e is:

$$P\left(\omega_i = 1 \middle| \eta_i^e\right) = P\left(H_i > C - J\eta_i^e\right) \qquad (4)$$

2.2. Individual interactions in the neighbourhood as a population game

The interest of studying such idiosyncratic (exogenous) heterogeneity becomes clearer if one reinterprets the GNP model within a game theoretic framework. Each agent i has only *two possible strategies*: to adopt $\omega_i = 1$ or not $\omega_i = 0$. In the following, we assume agents have myopic expectations about the behaviour of their neighbours: $\tilde{\omega}_{-i}(t) = \omega_{-i}(t-1) \equiv \omega_{-i}$, then $\eta_i^e(t) = \eta_i(t)$. The best response of an agent playing against its neighbours is formally equivalent to that of an agent playing against a *Neighbourhood Representative Player* (*NR*) (Phan and Pajot 2006, Phan and Shemeshenko, 2007). NR player in turn plays a *mixed strategy* $\omega_{nr} = \eta_i \in [0,1]$ In the present case of *finite neighbourhood local interaction*, ω_{nr} takes its value in a discrete subset of [0, 1]. For example for $N_{\vartheta i} = 2$, we have $\omega_{nr} = \eta_i \in \{0, 1/2, 1\}$ and for $N_{\vartheta i} = 4$, we have $\omega_{nr} = \eta_i \in \{0, 1/4, 1/2, 3/4, 1\}$. The "normal form" payoff matrix G1 gives the total payoff for an agent i playing against this *fictitious* NR player. According to (Monderer and Shapley, 1996), the best-reply sets and dominance-orderings of the game G1 are unaffected if a constant term is added to a column (i.e. $C - H_i$). The coordination game matrix G2 in Table 1.b is said to be "best reply equivalent" to the matrix G1 of Table 1.a. However, the values in G2 do not indicate the cumulated payoffs, contrary to the value in G1, but are a direct measure of the cost - the risk in the sense of (Harsanyi and Selten, 1988) - of a unilateral deviation from the coordinated solution ($\omega_i = \omega_{nr}$) in the case of the pure strategy framework.

Table 1. Payoff matrix for an agent i and best reply equivalent potential game

a - game G1	$\omega_{nr} = 0$	$\omega_{nr} = 1$
$\omega_i = 0$	0	0
$\omega_i = 1$	$H_i - C$	$H_i - C + J$

b - game G2	$\omega_{nr} = 0$	$\omega_{nr} = 1$
$\omega_i = 0$	$C - H_i$	0
$\omega_i = 1$	0	$H_i - C + J$

Player i in rows, fictitious NR Player - indexed nr - in columns

Figure 1.a presents a (symmetric triangular) pdf distribution and related best reply for a given cost C and a particular value of the IWA. If $C - J > H_i$, then *never adopt* ($\omega_i = 0$) is the strictly dominant strategy for all possible values of η_i (agents of the type (0) in the light grey zone on the left). If $H_i > C$, *always adopt* ($\omega_i = 1$) is the strictly dominant strategy for all possible values of η_i^e (agents of the type (1) in the dark grey zone on the right)/ If $C > H_i > C - J$ then the agent's virtual surplus $V_i \equiv H_i - C + J\eta_i$ may be either positive or negative depending on the values of the rate of adoption within the neighbourhood η_i. These agents are *conditional adopters* and said to be of the type (2). Within these agents, only those with $V_i > 0$ will adopt thanks to the social influence (hashed region). The relevant economic cases are the ones with (at least some) agents of type (2).

Figure 1.b exhibits a distribution of agents' type in the space (J, $H - C$) for the symmetric triangular distribution on [-a,a]. In the south-west light grey zone there are only agents of type (0), while in the north-dark-zone there are only agents of type (1). In the white zone there is a mixture of at least 2 types of agents, with necessarily some agents of type (2). If $H - C > a - J$, there is no agents of type (0). Conversely in the south zone, where $H - C < -a$, there is no agent of type (1) If both conditions hold then *all agents* are of type (2), corresponding to the hashed triangular zone in the east on figure 1.b. This implies a sufficiently strength intensity of social effect, with respect to the dispersion of the preferences, that needs to be relatively moderate: $J > 2a$.

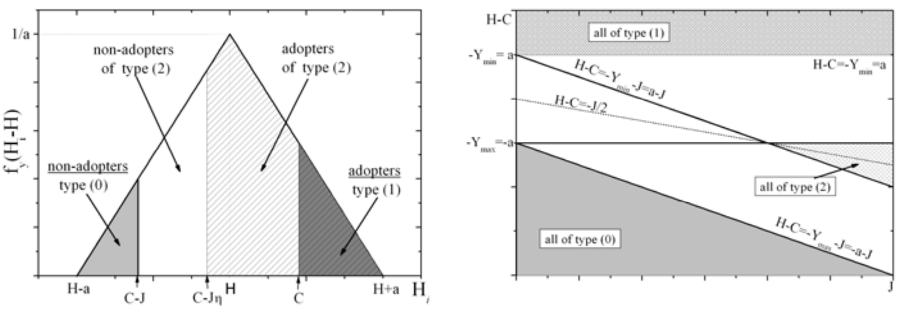

Fig. 1. Distribution of agents with respect to their type, on the pdf and in the space (J, H – C) for the symmetric triangular distribution on the interval [-a, a].
Source: (Phan and Semeshenko 2007)

In the case of global social influence (full connectivity) and bounded distribution of IWA, dominance-ordering analysis allows us to predict the issues of some classic configurations (i.e. symmetric Nash equilibrium), where all agents have the same structure of best reply. But this may be done for only some special cases. In more general situations, tools from statistical mechanics are necessary. In the case of local neighbourhood studied here, any simple result of that kind is available and probabilistic approach or numerical simulations are required. Within an approach of ACE as *complement* of traditional mathematical models, we compare in the two next sections results from the probabilistic approach with *infinite size* population with results from agent based simulation *for finite size* population.

3. Collective Behaviour, hysteresis and local frozen domains with local externality: probabilistic approach for *infinite size* population and simulation approach for *finite size* population.

As suggested before, the GNP model, as a socio-economic version of Quenched RFIM model, has some significantly different properties with respect to the BD model. Firstly, In the case of a change in the external field (i.e. cost variation) a particular equilibrium depends on the previous equilibrium, but does not depend on the order in which the agents change their behaviour (i.e. adoption or not) during the avalanche. In other words, from the simulation point of view, both parallel and sequential updating drive to the same equilibrium. Secondly, the interesting property of Sethna's inner hysteresis phenomenon (Sethna *et al.,* 1993) can be observed. These results from the *return point memory effect*: starting from a given equilibrium, if we change the cost by a given value and reverse the change by the same value, the system remembers its former state and returns exactly to the equilibrium point of departure. The corresponding trajectory is called “inner loop” or “minor hysteresis” (Sethna *et al.* 2005). Finally, in the special case where we change the cost monotonically for a homogeneous state (everybody adopts or nobody adopts) the final equilibrium does not depend on the rate of variation in cost. A dramatic change from C1 to C2 or a succession of smaller monotonic changes from C1 to C2 drives to the same state. In this section, we experiment the effect of local social influence in discrete choice adoption process based on the GNP model by means of finite size population, agent-based simulation on the multi-agent platform “Moduleco-Madkit” (Gutknecht and Ferber, 2000; Phan, 2004). Section 3.1. and 3.2 compare analytical results with simulated outcome in the case of the cyclical one-dimensional nearest neighbourhood network (circle). Section 3.2. is devoted to the calculus of the inner loop. Section 3.3. presents simulation outcome in the case of the cyclical two-dimensional regular network (von Neuman neighbourhood on a torus).

3.1. Starting from a homogeneous state without adoption to complete adoption and return: the larger hystersis loop.

Hysteresis within the Quenched RFIM is somewhat different in nature from the hysteresis used by economists that arise from a delayed response of a system (time lags) to change in the external parameter (here cost). First accounts of such difference are (Amable *et al.* 1994) about the wage-price spiral and zero-rot dynamics. Previous application of hysteresis in Quenched RFIM in socio-economic models are (Galam 1997: Phan *et al.* 2004). In the case of finite population, there are a very large number of equilibria and related thresholds between them. In this section, we use methodology and results from physics (Shukla 2000) established in the ferromagnetic case (J > 0) for one dimensional, nearest neighbourhood, cyclical and infinite size network. In that case, the conditional probability of adoption of equation (4) can be expressed in a finite number of occurrences (Table 2 and Fig. 2)

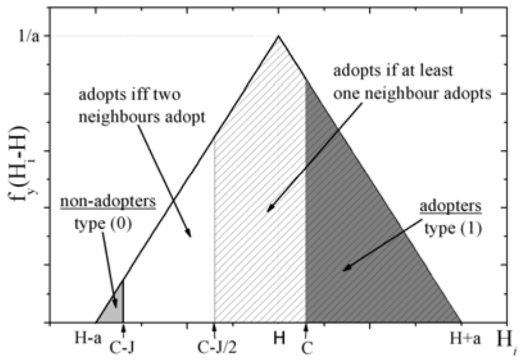

Fig 2. Agent's choice with respect to their IWA and neighbourhood state (symmetric triangular distribution).

$$P_0 = P\left(H_i > C\right) = P(\text{type } 1, \omega_i{=}1 \text{ if } \eta = 0)$$
$$P_1 = P\left(H_i > C - J/2\right) = P(\omega_i{=}1 \text{ if } \eta = 1/2)$$
$$P_2 = P\left(H_i > C - J\right) = P(\omega_i{=}1 \text{ if } \eta = 1)$$
$$P_1 - P_0\left(C\right) = \left(C > H_i > C - J/2\right) =$$
$$P(\text{to be of type } 2 \text{ AND } \omega_i{=}1 \text{ if } \eta = 1/2)$$
$$\text{with } P_i \equiv P_i(C)$$

Table 2 Probability of adoption for a given state of neighbourhood $\eta_i \in \{0, 1/2, 1\}$

For a given cost C, the probability of adoption of an agent is:

$$P\left(\omega_i = 1 \middle| C\right) = P\left(\eta_i = 1\right).P_2 + P\left(\eta_i = 1/2\right).P_1 + P\left(\eta_i = 0\right).P_0 \tag{5}$$

$$\text{with:} \quad \eta(C) = P\left(\omega_i = 1 \middle| C\right)$$
$$P\left(\eta_i = 1\right) = P\left(\omega_{i\pm1} = 1 \middle| C, \omega_i = 0\right)^2$$
$$P\left(\eta_i = 1/2\right) = 2.P\left(\omega_{i\pm1} = 1 \middle| C, \omega_i = 0\right) P\left(\omega_{i\pm1} = 0 \middle| C, \omega_i = 0\right)$$
$$P\left(\eta_i = 0\right) = P\left(\omega_{i\pm1} = 0 \middle| C, \omega_i = 0\right)^2$$

Where: $P\left(\omega_{i\pm1} = 1 \middle| \omega_i = 0\right) \equiv P^*(C)$ can be calculated exactly in the infinite case.

The probability that my neighbour adopts before me is equal first to P_0 (the probability of type (1), then adopts even if no neighbour has adopted before). One must add again the probability for my neighbour to be of type 2 but to adopt as soon as the next neighbour has adopted, since this next agent ($\omega_{1\pm 2}$) is of type (1). The corresponding join probability is equal to $[P_1 - P_0]P_0$. At the level 3, one must add again the probability for my neighbour and the next agent to be of type 2 but to adopt as soon as the next neighbour has adopted, given the probability that this next agent ($\omega_{1\pm 3}$) is of type (1). This join probability is equal to $[P_1 - P_0]^2 P_0$, and so on... Summing over all cases:

$$P^*(C) \equiv P\left(\omega_{i\pm 1} = 1 \middle| C, \omega_i = 0\right) = \lim_{m\to\infty} P_0 \sum_{k=0}^{m} \left[P_1 - P_0\right]^k = \frac{P_0}{1 - \left[P_1 - P_0\right]} \quad (6)$$

$$\left[1 - P^*(C)\right] \equiv P\left(\omega_{i\pm 1} = 0 \middle| C, \omega_i = 0\right) = \frac{1 - P_1}{1 - \left[P_1 - P_0\right]}$$

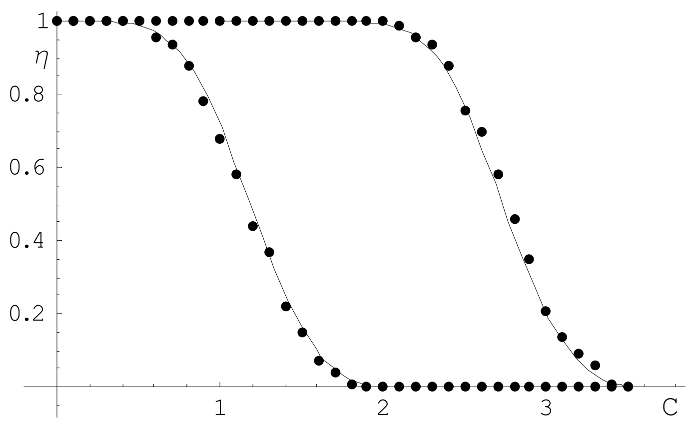

Fig. 3: Theoretic and simulated values (dot) for the cost-down branch of the main hystersis for: $N = 1156$ agents, $N_g = 2$ (circle); $J = 4$: H=0;

Through equations (5) and (6), the global equilibrium rate of adoption in the population for a giving cost C is equal to the probability of adoption of an agent taken at random within a symmetric triangular distribution of IWA:

$$\eta_+(C) = P\left(\omega_i = 1 \middle| C\right) \quad (7)$$

The upper half branch of the main hysteresis, for decreasing C from complete adoption to zero can be obtained by symmetry: $\eta_-(C) = -\eta_+(-C)$. Figure 3 provides a comparison between these theoretic values of the main hysteresis and the simulated ones, based in finite population experiments (here 1156 agents).

3.2. The inner hysteresis loop: reversing the Cost from an arbitrary point on the exterior loop.

In the limit of quasi-static driving (the change in prices remains constant within an avalanche), starting from a point on the upstream trajectory (grey) for η = 40%, and C = 1.25 a backtracking increase in cost C induces a less than proportional decrease (avalanche) in the number of customers (black curve, upper inner loop) until C = 2.49 and η = 30%. Then after reversing the cost changes at C = 2.49, as the cost decreases back to the initial value (grey curve, lower inner loop) the system returns precisely to the same state from which it left the outer loop (C = 1.25, η = 40%). The inner loop can also go from a branch of the main hysteresis loop to the other. For example, starting at C = 1 and η = 68%, a backtracking increase in cost C induces a cross-trajectory between the upstream and the downstream branch of the main hysteresis loop. This cross-trajectory finishes at C = 2,93 and η = 30%, when the equilibrium points are those of the main hysteresis. As established analytically, that confirms there is a multiplicity of equilibria, depending on the previous state of the system (*path-dependence*). Figure 3.b exhibits separated homogeneous domains (or cluster) in the network, due to the dominance of positive or negative effects of social influence as well as a particular distribution of heterogeneous IWA, enforced by both locality and finite size effect.

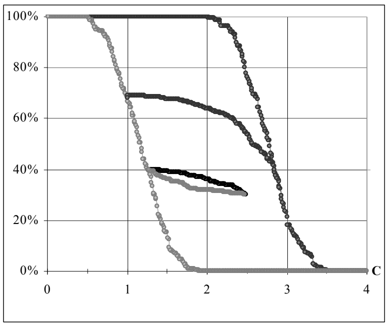

Fig. 4.a Sethna's inner hysteresis J=4, N = 2 (circle)

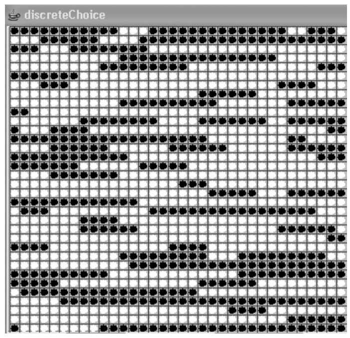

Fig. 4.b homogeneous domains (1D-clusters) within the network for η=40% C=1,27

As previously, it is possible to provide some hints to calculate the probability of adoption starting from an arbitrary point of the exterior loop. The method used here follows the work of (Shukla 2000). For the reversing formula and complete calculations in the case of the symmetric triangular pdf, see the long version of this work (to be presented at CEF 2007). This calculus is more difficult than in the previous case, because the choice of adoption depends now in a non trivial way on the rate of adoption in the neighbourhood, which depends itself directly or indirectly on the state of the other agents over the network. The probability of adoption between the two branches of the external hysteresis is conditional to the cost C for which the backtrack starts. These analytical results fit correctly the numerical simulations in the case of finite population experiments (see long version for CEF 2007)). For a given cost *C*', with backtracking at *C*, the probability of adoption of an agent is:

$$P\left(\omega_i = 1\middle|, C'C\right) = P\left(\omega_i = 1, C\right) - Q_2\left(C', C\right) - Q_1\left(C', C\right) - Q_0\left(C', C\right) \qquad (8)$$

$$with: \quad \eta(C', C) = P\left(\omega_i = 1 \middle| C', C\right)$$

$$Q_2\left(C', C\right) = P\left(C\right)^2 \left(P_2(C) - P_2(C')\right)$$

$$Q_1\left(C', C\right) = 2.P^*\left(C\right)\left[Q_a\left(C', C\right) + Q_b\left(C', C\right)\right]\left(P_1(C) - P_1(C')\right)$$

$$Q_0\left(C', C\right) = \left[Q_a\left(C', C\right) + Q_b\left(C', C\right)\right]^2 \left(P_0(C) - P_0(C')\right)$$

$$Q_a\left(C', C\right) = \frac{P^*\left(C\right)\left(1 - P_2(C)\right) + \left[1 - P^*\left(C\right)\right]\left(1 - P_1(C)\right)}{1 - \left(P_1(C) - P_1(C')\right)}$$

$$Q_b\left(C', C\right) = \frac{P^*\left(C\right)\left(P_2(C) - P_2(C')\right)}{1 - \left(P_1(C) - P_1(C')\right)}$$

3.3. The two-dimensional von Neuman neighbourhood network (Torus)

There is no analytical result at this time for the two-dimensional, cyclical network with von Neuman neighbourhood (Torus). But the example of figure 5 suggests that both Sethna's inner loop and homogeneous domains remain quite similar.

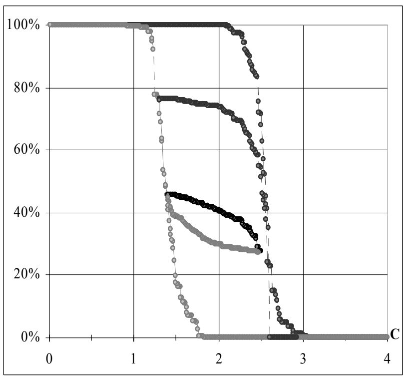

Fig. 5.a Sethna's inner hysteresis J=4, N = 4 (Torus)

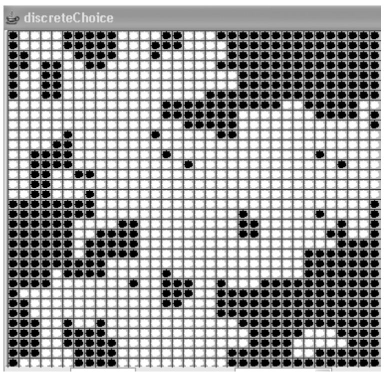

Fig. 5.b homogeneous domains (2D-clusters) within the for η = 40% C=1.41

5. Conclusion

Using methodology within a framework of statistical physics, we illustrated the stationary properties for particular cases of symmetric triangular distribution of IWA under local interactions. Results of the simulation allow us to observe numerous complex configurations on the adoption side, such as hysteresis, and Sethna's inner-loop hysteresis. This complex social phenomenon depends significantly on the structure and parameters of the relevant network. Finally, the last section opens the question of *finite size effects*, also adressed by (Glaeser and Sheinkman, 2002; Krauth, 2006) among others. Such preliminary results in the case of simple, regular network suggest new fields of investigation, as opposed to a standard focus on conditions of uniqueness of equilibrium, under a "moderate social influence" assumption (Glaeser and Scheinkman, 2002). It would be interesting in the future to compare more systematically the analytical predictions against the simulation results and to study the statistical properties of such a phenomenon for different values of J and different network's structure.

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