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Contents

1 Social Interactions in Economic Theory: an Insight from Statistical Mechanics

Denis Phan[1], Mirta B. Gordon[2], and Jean-Pierre Nadal[3]

[1] ENST de Bretagne and ICI Université de Bretagne Occidentale
[2] Laboratoire Leibniz-IMAG, 46, Av. Félix Viallet, 38031 Grenoble Cedex 1, France
[3] Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris cedex 05, France

Abstract. This Chapter extends some economic models that take advantage of a formalism inspired from statistical mechanics to account for social influence in individual decisions. Starting with a framework suggested by Durlauf, Blume and Brock, we introduce three classes of models shifting progressively from rational towards adaptive expectations. We discuss the risk and opportunity of transposing the tools, methods and concepts from statistical mechanics to economics. We also analyze some issues seldom addressed, such as a comparison between two models of heterogeneous idiosyncratic preferences, corresponding to cases with quenched and annealed disorder in statistical physics, respectively.

1.1 Introduction

The present Chapter provides a bird's eye view of some literature in economics that exhibits strong links with statistical mechanics (introduced in the Chapters **??** by Gordon and **??** by Galam, this book). The aim of this literature is to determine aggregate behaviours based on models where individual decisions are subject to social influence or social interaction effects.

Föllmer [29] was among the first to realize that the social effects in microeconomic models could be introduced through the Gibbs' probabilistic framework. His paper determined conditions for the existence of a price system in a pure exchange economy with random interdependant preferences. Almost at the same time, the question of the influence of neighbourhood on behaviour and micro-macro tradeoff was explored in the field of social sciences (see [65] for an early model of binary choices with externalities and Phan, chap. **??**, this book, for the Shelling's model of segregation) In 1980 Kindermann and Snell [44] suggest possible applications of Markov Random Fields [43] within the growing field of social networks (for recent survey, see e.g Wasserman and Faust [70]). In 1982, an economist, Beth Allen [1,2] proposed two models of stochastic macro-dynamics based on local interactions. On the one hand, she deals with the diffusion of new technologies with externalities. On the other hand, she considers the role of transmission of information among individuals in a stochastic model with local interactions. Earlier attempts to model social and economic phenomena by physicists, date back at least to the '70 (Weidlich [71,72]). More recently, we can mention works by Galam et al [34,31]. Most of these papers are not well known within economics.

In the middle of the 80's, Kirman [45,51] suggests the use of stochastic graph theory in order to take into account the local communications between agents within the markets. The real take-off for the statistical mechanics models of locally interacting agents written by economists began in the 90's in the USA. The main contributor was Stephen Durlauf [20–23]. See also among others, Ioannides [40] and Glaeser et al. [36,37]. On the heterodox economic side, Markov Random Fields have also been introduced in innovation economics, by Dalle [19] among others. During this period both Brocke [12] and Blume [7,8] introduced explicit links between game theory and statistical mechanics In the case of social influence, the inferential approach raises the question of a common reference. Mansky [56] examines the "reflection problem" that arises when the researcher tries to infer whether the average behaviour of a group influences the individual behaviour of the members of the group by an endogenous social effect. Given the reflection problem, Brocke [12] proposes a behavioural foundation of the common physicists' simplification, the so-called "mean field approximation" (see Galam, Chap. **??**, this book). Instead of building his expectation through pairwise interactions with his neighbours, each agent bases his expectation on the average behaviour. The approach through this class of models has been reviewed in numerous synthesis papers by Durlauf and co-workers, Blume and Brock [9,13,24,26], hereafter referred to as DBB. Useful technical reviews of this literature are also provided by Ioannides [41]. Kirman ([46,48] and Chapter **??**, this book) provides complementary and stimulating discussions about the role of interactions and networks in economics.

In recent years, a growing field of so-called "econophysics" has developed within the physicists' community (see for instance [28,54,57]). Within the important part of this literature devoted to mathematical finance, several models address the effect on the market of interactions between agents [57,15,14].

Within a unified formal framework with notations close both to Gordon (Chap. **??**, this book) and Durlauf [26], this Chapter presents a class of models that generalize standard economic models of individual discrete choices by including the social influence explicitly.

In section 1.2 we review the models proposed by DBB. This approach has helped to disseminate the tools of statistical mechanics into the field of economic models. It is close to both the game theoretic framework and rational expectations theory. Section 1.3 review our own results [59,64,69] on a model of a discrete choice monopolist market with demand externality. The model remains close to the DBB framework, but has a simpler expectation structure: no expectations at all on the demand side (myopic customers have no strategic behaviour) and adaptive expectations on the supply side. Considering the optimal price for a monopolist, we exhibit an interesting phenomenon, analogous to a *first order phase transition* in physics. Phan (Chap. **??**, this book), and Weisbuch and Stauffer (Chap. **??**, this book), discuss dynamical aspects of such models. The models considered in these sections postulate a logit distribution of individual payoffs, widely used by economists to formalize individual decisions. Section 1.4 considers the search process over heterogeneous sellers, initially developed for the modeling of a wholesale fish market [49,60,73], with the aim of showing that the logit choice function may be deduced from an exploration-exploitation compromise between the immediate maximization of the surplus and the search for information about the market. The conclusion of the Chapter presents a short summary and opens some new perspectives.

1.2 Discrete choice with social interactions (I): Individual strategic behavior and rational expectations

1.2.1 Basic model of individual behaviour

The DBB basic model considers a population of N agents. Each agent has to make a binary choice s_i in the set {-1, 1}. Agents are assumed to maximise an expected utility:

$$V_i\left(s_i\right) = \left(h_i + \epsilon_i\right) s_i + \hat{E}_i[S(s_i, \mathbf{s}_{-i})] \tag{1.1}$$

Specification (1.1) embodies both a “private” and a “social” component. The private component includes a deterministic part: $h_i s_i$ and a stochastic part: $\epsilon_i s_i$, where ϵ_i is a random variable independent of the agent’s choice. This random variable can be understood as representing some kind of boundedness in the agent’s rationality. It can also be interpreted as the result of external shocks imposed on the agents. If the law of ϵ_i has zero mean, $h_i s_i$ can be interpreted as the expected utility of agent i, in the absence of social effects. The social component $S(s_i, \mathbf{s}_{-i})$ takes into account the interactive dimension of the decision process, i.e. the social effect on the utility of agent i due to the behaviour of the other agents. $\mathbf{s}_{-i}$ denotes the choice vector of the neighbours of agent i, a subset of agents denoted ϑ_i. More specifically, DBB assume information asymmetry: each agent i knows his own choice, but has to make assumptions about the behaviour of the agents in his “neighbourhood” ϑ_i. The quantity $\hat{E}_i[.]$ denotes agent’s i belief, or subjective expectation, about the effect of such neighbourhood behaviour on his own utility.

The maximisation of the N equations (1.1) should be done simultaneously by all the agents, $i = 1, ..., N$. In these equations, classical rationality is relaxed at two different levels. The “noise” ϵ_i introduces some indeterminacy into the private component of the utility. The other source of boundedness in rationality may arise when the subjective expectations $\hat{E}_i[.]$ are inconsistent with the *a posteriori* probability distribution of choices.

The choice that maximizes (1.1) is $s_i = 1$ if $h_i + \epsilon_i + \hat{E}_i[S(+1, \mathbf{s}_{-i})] \geq -h_i - \epsilon_i + \hat{E}_i[S(-1, \mathbf{s}_{-i})]$, that is,

$$s_i = +1 \quad \text{if} \quad -\epsilon_i \leq h_i + \frac{\hat{E}_i[S(+1, \mathbf{s}_{-i})] - \hat{E}_i[S(-1, \mathbf{s}_{-i})]}{2} \tag{1.2}$$

Otherwise $s_i = -1$. Thus, in this model there are two different levels of indeterminacy, since the choice depends on the values taken by the random variable ϵ_i, and on the subjective beliefs. The latter depend on the utilities, h_k and ϵ_k, as well as on the beliefs of the neighbours of i. None of these quantities are available to i. Even if there were no random payoffs, that is, if $\epsilon_i = 0$ for all i, the agents ignore the payoffs and beliefs of their neighbours.

Knowing the probabilities $P_i(\epsilon_i)$, it is possible to calculate the expectation of the agents choices as a function of their beliefs, since

$$P_i(s_i = +1 | \hat{E}_i[S(+1, \mathbf{s}_{-i})]; \hat{E}_i[S(-1, \mathbf{s}_{-i})]) = \\ P_i(-\epsilon_i \leq h_i + \frac{\hat{E}_i[S(+1, \mathbf{s}_{-i})] - \hat{E}_i[S(-1, \mathbf{s}_{-i})]}{2}). \tag{1.3}$$

In the case of rational expectations [12], the subjective expectations $\hat{E}_i[.]$ coincide with the true *mathematical* expectations $E[.]$. This imposes a non-trivial self-consistence condition, further discussed in section 1.2.2.

Let us denote J_{ik} the (marginal) social influence on agent i, that is, the incidence on individual i of the decision of agent $k \in \vartheta(i)$. Formally, DBB define J_{ik} as the second order cross-derivative of $S(s_i, \mathbf{s}_{-i})$ with respect to s_i and s_k (this definition applies to the restriction of a continuous function $S(s_i, \mathbf{s}_{-i})$ to binary arguments, s_i and s_k):

$$\frac{\partial^2 S(s_i, \mathbf{s}_{-i})}{\partial s_i \partial s_k} = J_{ik}, \tag{1.4}$$

and assume a positive influence, i.e. strategic complementarity [17,18]. There are at least two simple specifications for S that satisfy condition (1.4). On the one hand, one can assume a negative quadratic conformity effect [5]:

$$S(s_i, \mathbf{s}_{-i}) = -\sum_{k \in \vartheta_i} \frac{J_{ik}}{2} (s_i - s_k)^2 \tag{1.5}$$

In this case, when agents are disconnected ($J_{ik} = 0$) or when choices are similar ($s_i = s_k$), the social effect vanishes. As soon as an agent's decision differs from that of one of his neighbours, there is a local effect of negative value. As a consequence, $2J_{ik}$ can be interpreted as the loss of agent i if his own choice s_i does not agree with the choice of his neighbour k. If $J_{ik} = J_{ki}$ there is reciprocity.

Another specification currently considered in the literature is the following positive and multiplicative expression:

$$S(s_i, \mathbf{s}_{-i}) = s_i \sum_{k \in \vartheta_i} J_{ik} s_k. \tag{1.6}$$

Specifications (1.5) and (1.6) both lead to the same optimization problem for utility (1.1): since $s_i^2 = s_k^2 = 1$ for all i, k, the quadratic conformity effect (1.5) may be written as follows:

$$\begin{aligned} S(s_i, \mathbf{s}_{-i}) &= + \sum_{k\in\vartheta_i} J_{ik} s_i s_k - \sum_{k\in\vartheta_i} \frac{J_{ik}}{2} \left(s_i^2 + s_k^2\right) \\ &= s_i \sum_{k\in\vartheta_i} J_{ik} s_k - \sum_{k\in\vartheta_i} J_{ik}. \end{aligned} \tag{1.7}$$

which differs from (1.6) by an irrelevant constant value. Hereafter we use formulation (1.6).

We further assume that each agent i knows precisely his marginal losses J_{ik} due to non conformity, and has only to estimate his neighbours' choices in (1.6):

$$\hat{E}_i[S\left(s_i, \mathbf{s}_{-i}\right)] = \hat{E}_i[s_i \sum_{k\in\vartheta_i} J_{ik} s_k] = s_i \sum_{k\in\vartheta_i} J_{ik} \hat{E}_i[s_k]. \tag{1.8}$$

Then, the subjective expectation of agent i about the social effects is completely determined by his beliefs $\hat{E}_i[s_k]$ about his neighbours' choices. Introducing (1.8) into (1.1), the maximization of the utility can be written as follows:

$$\max_{s_i\in\{-1,1\}} V_i(s_i) = \max_{s_i\in\{-1,1\}} s_i(h_i + \epsilon_i + \sum_{k\in\vartheta_i} J_{ik}\hat{E}_i[s_k]) \tag{1.9}$$

Introducing these asumptions into (1.3), we obtain

$$P_i(s_i = +1|\{J_{ik};\, \hat{E}_i[s_k]\}_{k\in\vartheta_i}\}) = P_i(-\epsilon_i \le h_i + \sum_{k\in\vartheta_i} J_{ik}\hat{E}_i[s_k]). \tag{1.10}$$

Let us define the "field" variables

$$\hat{z}_i = h_i + \sum_{k\in\vartheta_i} J_{ik}\hat{E}_i[s_k] \tag{1.11}$$

where the hat indicates that these depend on the subjective beliefs of agent i.

A formal analogy between the present model and the Ising model of statistical mechanics appears if we assume, following DBB, logistic distributions for the agents' random payoffs ϵ_i:

$$P_i(-\epsilon_i \le z) = \frac{1}{1+\exp(-2\beta_i z)}, \tag{1.12}$$

The parameters β_i control the width of the distributions. Notice that a logistic distribution of parameter β has a sigmoidal shape very similar to that of an error function of standard deviation

$$\sigma \sim \frac{1}{\beta}, \tag{1.13}$$

and presents the advantage of having an analytical expression.

With logit assumption (1.12) , probability (1.10) becomes

$$P_i(s_i = \pm 1 | \beta_i,\, \hat{z}_i) = \frac{\exp(\pm \beta_i\, \hat{z}_i)}{\exp(\beta_i\, \hat{z}_i) + \exp(-\beta_i\, \hat{z}_i)} \tag{1.14}$$

From (1.14) we obtain the *mathematical* conditional *expectation* of agent i's choice, given his own expectations about the other agents' behaviours:

$$E[s_i | \beta_i\, \hat{z}_i] = \frac{\exp(\beta_i\, \hat{z}_i) - \exp(-\beta_i\, \hat{z}_i)}{\exp(\beta_i\, \hat{z}_i) + \exp(-\beta_i\, \hat{z}_i)} = \tanh(\beta_i\, \hat{z}_i). \tag{1.15}$$

Notice that in (1.15), we write $E[s_i|\beta_i\, \hat{z}_i]$ without a hat, to represent the *true* mathematical expectation of agent i's choice, *given the field* $\hat{z}_i$. To calculate this expectation, it is necessary to know the distribution of the random payoffs ϵ_i. In principle, this knowledge is not available to the agents, who may assume probability laws $\hat{P}_i(s_i = \pm 1|\beta_i,\, \hat{z}_i)$ which may be very different from (1.14).

In the following, let us assume that probabilities (1.14) are known to the agents. Then (1.15) only depends on the product $\beta_i \hat{z}_i$, where the expectations about the neighbours' choices that enter the definition of $\hat{z}_i$ are subjective, and in general are not equal to the mathematical expectations of the choices of agents $k \in \vartheta_i$. Up to now, the way the agents determine their subjective expectations $\hat{E}_i[s_k]$ entering the definition of $\hat{z}_i$ have not been specified. They can, at least in principle, be arbitrary.

At this stage the link with statistical physics is as follows. For any given set of expected utilities $\hat{\boldsymbol{z}} = \{\hat{z}_i, i = 1, ..., N\}$, agents choices are independent random variables. Their joint probability is just the product of probabilities (1.14), which is formally equal to the Gibbs Law for a system of *non-interacting* Ising spins (see equation (42), section 4 of the Chapter **??** by Gordon, and the Chapter **??** by Galam, this book), where each spin is in a *local* magnetic field $\hat{z}_i$, at a local temperature $T_i = 1/\beta_i$. Accordingly, the (true) expectation (1.15) of agent i's choice corresponds to the average magnetization of the spin i in a field $\hat{z}_i$.

Furthermore, if $\beta_i = \beta$ is the same for all agents, all the spins are at the (same) reservoir's temperature $T = 1/\beta$; we may drop the subscripts i in the β_i and in the mathematical expectations, and write $E[s_i|\beta\hat{z}_i]$. When T goes to zero, which corresponds to the limit $\beta \to \infty$, the expected (subjective) utilities are maximized by choices $s_i = \text{sign}[\hat{z}_i]$. If β is finite, the probabilities of choices given by (1.14) with $\beta_i = \beta$ lead to the average choice (1.15). In the extreme case of a very high temperature (small β) the random payoffs may be so large that the two possible choices have almost the same probability, the corresponding mathematical expectations being close to zero.

1.2.2 Equilibrium with rational expectations

Within the special (neo-classical) approach of rational expectations, all the agents have the same, rational, behaviour. This results from the preceding formulation, if one assumes that the expectations of each agent i about the choices of his neighbours, $\hat{E}_i[s_k]$ in the right hand side of (1.9), are consistent with the true mathematical expectations of these neighbours *deduced using probabilities* (1.14). That is, for all i and all k, rationality requires:

$$\hat{E}_i[s_k] = E[s_k|\beta_k \hat{z}_k], \tag{1.16}$$

for all $k \in \vartheta_i$, where $E[s_k|\beta_k\, \hat{z}_k]$ stands for the true mathematical expectation of agent k's choice (see equation (1.15)).

Depending on the kind of neighbourhood, conditions (1.16) may be more or less easy to satisfy. In the simplest case where the agents' utilities do not depend on social effects [39], that is $J_{ik} = 0$ for all i, k, each agent maximizes his own utility independently of the others, and the probability of choices is given by (1.14) with $\hat{z}_i = h_i$. Another treatable case is that of complete asymmetry, in which the neighbours of agent i do not suffer from his social influence; that is, if whenever $J_{ik} \neq 0$, then $J_{ki} = 0$. Under such conditions, if the choices are made in an adequate (temporal) order, the agents may make their decisions with complete knowledge. This is not possible in the general case, in which if $J_{ik} \neq 0$ (the choice of agent k has an effect on the utility of i), then $J_{ki} \neq 0$. In this case, equations (1.16) become entangled.

In the absence of selfconsistency, the $\hat{E}_i[s_k]$ are nothing but numbers (they may even not be mathematical expectations over any specific distribution at all): they are the numerical value of $E[s_k]$ assumed by agent i. The choice of i depends on these numbers through the fields $\hat{z}_i = h_i + \sum_{k \in \vartheta_i} J_{ik} \hat{E}_i[s_k]$ and we have (see definition (1.15)) $E[s_i|\beta_i \hat{z}_i] = \tanh[\beta_i \hat{z}_i]$.

The selfconsistency conditions for rationality obtained from (1.15) and (1.16) require that the numbers $\hat{E}_i[s_k]$ satisfy

$$\hat{E}_i[s_k] = \tanh[\beta_k(h_k + \sum_{l \in \vartheta_k} J_{kl} \hat{E}_k[s_l])] \tag{1.17}$$

meaning that the beliefs of agent i about his neighbours choices must coincide with his own expectations.

System (1.17) has at least one fixed point. If we assume that the agents' random payoffs all obey the same probability law, that is, $\beta_i = \beta$, and that there is perfect reciprocity in the social effects, that is, the interactions are symmetrical, $J_{ik} = J_{ki}$, then equations (1.17) coincide with the mean field equations of the Ising model *with interactions* J_{ik}, in local external fields h_i, at temperature β^{-1}. The solution is far from trivial: depending on the sign and range of the interactions J_{ik}, on the distribution of the local fields h_i, and the temperature, many different fixed points may exist.

In the extreme case where the agents' private utilities h_i are random, the present model is rather similar to the Ising Model in Random Fields, introduced in this book, Chap. **??** by Galam. Even if $h_i = h$ is the same for all the agents, a huge number of fixed points, of the order of 2^N, may exist under some conditions, like in the spin-glass models of physics.

An interesting simple case, considered in DBB, is that of the homogeneous system with local interactions, where all neighbourhoods ϑ_i have the same size n, the interaction parameters all have the same value J/n with $J > 0$, all the agents have equal private payoffs $h_i = h$, and the same β_i. In this case of perfect reciprocity and rationality, all the agents have the same expectations:

$$\hat{E}_i[s_k] = E[s|\beta\hat{z}_i] = m \tag{1.18}$$

for all i and all k. Then, the effect of the social influence represented by the sum $\sum_{l\in\vartheta_k} J_{kl}\hat{E}_k[s_l]$ in (1.17) reduces to $J\,m$ for all i and k. The expected value of the social choice is the solution of the single mean field equation of the Ising model:

$$m = \tanh(\beta(h + Jm)) \tag{1.19}$$

According to well known standard results in statistical mechanics [67], there exist either one or three solutions to (1.19), depending on the relative magnitudes of the private utility, the distribution width of the stochastic term and the intensity of the social effects. These results were summarised for economists in [13]:

i) if $h = 0$, then
  a) if $\beta J < 1$ then there is a single solution to (1.19), $m = 0$
  b) if $\beta J > 1$, besides the solution $m = 0$ there are two other solutions: a positive one ($m > 0$) and a negative one ($m < 0$).
ii) if $h \neq 0$, then
  a) if $\beta J < 1$, there is a single solution to (1.19), with $\mathrm{sign}(m) = \mathrm{sign}(h)$.
  b) if $\beta J > 1$ and $J > h > 0$, besides the solution with $\mathrm{sign}(m) = \mathrm{sign}(h)$, there are two other solutions, with $m < 0$.

Notice that the latter solutions, which only arise if the social effects are strong enough, correspond to average choices driven by the social component, which are in contradiction with the private component of the utility function. This situation may arise when the agent's utility is dominated by these social effects. Figures 1.1 represent the left right hand side of equation (1.19) as a function of h, for different values of the parameters. The solutions correspond of the intersections between these curves and the straight line $y = m$.

1.2.3 Economic Interpretation

The magnitude of J represents the weight of the social influence on the individual choice. In this model, the larger J, the stronger the influence of the others choices on each agent behaviour, and the weaker the relative weight of the private component, h. For large values of J, these social effects result in the emergence of a specific order, produced by the social interaction, in which all the agents make coordinated choices.

Assuming that every agent anticipates the same solution m, then each agent i can make his choice according to the same probability law $P(s_i|\beta\,\hat{z}_i =$

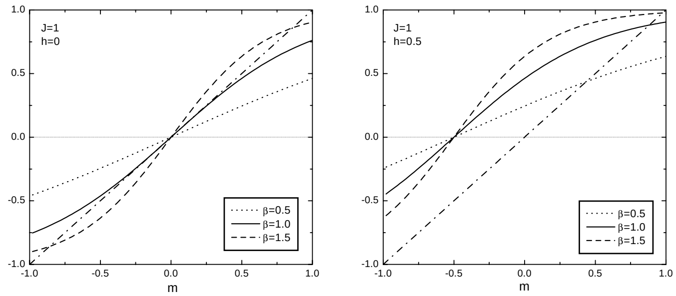

Fig. 1.1. Solutions of the Ising model with $J = 1$, for $h = 0$ and $h = 0.5$, and different values of the parameter β

$\beta(h + Jm))$ as usual within the framework of rational expectations. However, even in the case of homogeneous interactions, the possibility of having an equilibrium requires a strong hypothesis (i.e. all the agents know the probability distribution of the random payoffs, and the values of h and β). In this simple and idealized framework of perfect rational equilibrium, every agent, as well as the modeller, may compute the equilibrium parameter m. Thus, the equilibrium state is known *before* any agent has actually made his choice. However, this paradoxical conclusion may only be reached when there is a single equilibrium. For, when social influence matters, multiple equilibria may exist even with perfect knowledge, as seen above. In particular, one of the possible equilibria corresponds to individual choices that disagree with the individual preferences. This arises when the degree of conformism is so strong (J large) that the interactions may lead the mean social choice to differ from the one that would result without interactions. Thus, the knowledge of private individual preferences is not sufficient to predict the collective behaviour, as pointed out by Durlauf [26]: “When individual behaviour is driven by a desire to be similar to others, this does not provide any information on what they actually do; rather it merely implies that whatever behaviour occurs, there will be substantial within-group correlation due to the conformity effects”.

In order to select one equilibrium, and to characterize its properties, some collective dynamical process has to be taken into account. This requires leaving the strict perfect rational expectations framework. One interesting and already classical approach is through adaptive rationality, as in [38]. If the agents have to make choices repeatedly, the $\hat{E}_i[s_k]$ may be estimated using a probabilistic approach, with tools of statistical inference. These allow us to determine either the underlying probabilities $\hat{P}_i[s_k]$, or, directly, the expectations $\hat{E}_i[s_k]$. In such an adaptive framework, it is not necessary for the agents to know in advance the population parameters (e.g. fields h_i), and not even to anticipate the set of possible solutions: it is the collective dynamics that may allow the population to converge towards a possible equilibrium. Cases of adaptive rationality will be discussed in the following sections.

1.3 Discrete choice with social interactions (II): Market price and adaptive expectations

1.3.1 Monopoly market with homogeneous product

In this section we explore the effects of localised externalities (social influence) upon the properties of a market with discrete choices [3]. More specifically, we focus on the simplest case: a single homogeneous product and a single seller (the monopoly case). Following Kirman [46,47,50], the resulting market is viewed as a complex interactive system with a communication network. On the demand side, behaviour remains close to the DBB framework, but without expectations. Customers are assumed to be myopic and non strategic. On the supply side, the only cognitive agent in this process is the monopolist. In the general case, the local interactions produce complex phenomena, and the behaviours shift from rational towards adaptive expectations.

Within this framework, we discuss an issue not addressed by DBB: the comparison between two models of idiosyncratic heterogeneity. In one model, the payoffs are randomly chosen and remain fixed. In the other, the preferences of agents are assumed to fluctuate independently around a fixed (homogeneous) value. The former case is known by physicists as a model with *quenched disorder*, whereas the latter corresponds to an *annealed disorder*. In both cases we assume that the heterogeneous preferences of the agents are drawn from a same (logistic) distribution. The equilibrium states of the two models generally differ, except in the special case of homogeneous interactions with complete connectivity. In this special situation, which corresponds to the mean-field case in physics, the expected aggregate steady-state is the same in both models.

In the basic model [59,64,69], the agents have a classical linear willingness to pay function. Each agent i either buys one unit of a given good ($\omega_i = 1$) or does not buy it ($\omega_i = 0$). A rational agent chooses ω_i in order to maximize his *surplus function* V_i:

$$\max_{\omega_i \in \{0,1\}} V_i = \max_{\omega_i \in \{0,1\}} \omega_i (h_i + \epsilon_i + J_\vartheta \sum_{k \in \vartheta_i} \omega_k - p) \tag{1.20}$$

where h_i represents the idiosyncratic (deterministic) preference of the agent, ϵ_i is a random component that may temporarily modify this preference, and p the price of one unit.

The relation with Statistical Mechanics is completely transparent if we transform the variables $\omega_i \in \{0,1\}$ into $s_i = \pm 1$ through

$$\omega_i = \frac{1 + s_i}{2}, \tag{1.21}$$

which is nothing but a change in notation. After introducing (1.21) together with the definitions $\tilde{h}_i = h_i + \frac{1}{2} \sum_{k \in \vartheta_i} J_{ik}$ and $\tilde{J}_{ik} = \frac{1}{2} J_{ik}$ into (1.20), we obtain:

$$\tilde{V}(s_i) = \frac{s_i}{2}(\tilde{h}_i + \epsilon_i + \sum_{k \in \vartheta_i} \tilde{J}_{ik} s_k - p). \tag{1.22}$$

All the expressions in the present Chapter can be put in terms of either s_i or ω_i using transformation (1.21). In the following we will make use of the encoding $\omega \in \{0, 1\}$.

TP versus McF discrete choice models. Within this basic framework, Nadal et al [59] compare two extreme special cases of the discrete choice model. Following the typology proposed by Anderson et al. [3]), they distinguish a "psychological" and an "economic" approach to the individual's choice. Within the psychological perspective (Thurstone [68]), the utility has a *stochastic* aspect because "there are some qualitative fluctuations from one occasion to the next ... for a given stimulus" (this point of view will be referred to hereafter as the TP-case). On the contrary, for McFadden [58] each agent has a *deterministic* willingness to pay, that may differ from one agent to the other. The seller (in a "risky" situation) cannot observe each specific idiosyncratic willingness to pay, but knows its statistical distribution over the population (we call this perspective the McF-case). Accordingly, the "TP" and "McF" perspectives only differ by the nature of the individuals' willingness to pay.

In both cases, for simplicity one assumes homogeneous local interactions and identical neighbourhood structures ϑ, of size n, for all the agents,

$$J_{ik} = J_\vartheta \equiv J/n > 0. \tag{1.23}$$

In the McF model, the agents differ by their "private" idiosyncratic terms h_i. These are randomly distributed over the agents, but remain fixed during the period under consideration. The temporal variations ϵ_i are strictly zero. For physicists, this model with fixed heterogeneity belongs to the class of *quenched disorder* models (the values h_i are equivalent to random, time-independent, local fields). More precisely, the McF model is equivalent to a *random field Ising model* (RFIM), at zero temperature (deterministic dynamics). Since we assumed ferromagnetic couplings (that is, the interaction J between Ising spins is positive), the spins s_i tend to take all the same value. This "agreement" may be broken by the influence of the heterogeneous external fields h_i. Due to the random distribution of h_i over the network of agents, the resulting organisation is complex. In the following, we introduce the following notation: $h_i = h + \theta_i$, and we assume that the θ_i are logistically distributed with zero mean and variance $\sigma^2 = \pi^2/(3\beta^2)$:

$$\lim_{N\to\infty} \sum_i \theta_i = 0 \Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_i h_i = h \tag{1.24}$$

In the TP model the agents all have the same deterministic component h, but have an additive random idiosyncratic characteristic, ϵ_i. The ϵ_i are i.i.d. random variables of zero mean, that are refreshed at each time step. In physics this problem corresponds to a case of *annealed disorder*. The time varying random idosyncratic component is equivalent to having a stochastic dynamics. Agent i decides *to buy* according to the *logit choice function* , with conditional probability

$$P(\omega_i = 1|z_i) = P(\epsilon_i > z_i) = 1 - F(z_i) \equiv \frac{1}{1+\exp(\beta z_i)},$$
$$\text{with} \quad z_i = p - h - J_\vartheta \sum_{k \in \vartheta_i} \omega_k. \tag{1.25}$$

This model is then equivalent to an Ising model in a uniform (non random) external field $h - p$, at temperature $T = 1/\beta$. Expression (1.25) differs from (1.12) by a factor 2 in the exponent due to the factor 1/2 in front of equation (1.22).

From the physicist's point of view, McF and TP models are quite different: random field and zero temperature in the former, uniform field and non zero temperature in the latter. The properties of disordered systems have been and still are the subject of numerous studies in statistical physics (for an introduction and some references, see the Chapter **??** by S. Galam, this book). An important result is that quenched and annealed disorder can lead to very different behaviours.

TP model is well understood. Even if a general analytical solution of the optimization problem (1.20) does not exist, the mean field analysis gives approximate results that become exact in the limiting situation where every agent is connected (i.e. is a neighbour) to every other agent. The exact analysis of the case where the agents are placed on a 2-dimensional square lattice, and has four neighbours, has been a *tour de force* due to Onsager [61]. On the contrary, the properties of the McF model are not yet fully understood. However, a number of important results have been published in the physics literature since the first studies of the RFIM by Aharoni and Galam [32,33] (see also [30], [66]). Several variants of the RFIM have already been used in the context of socio-economic modeling ([34,62], Weisbuch and Stauffer, this Book).

From the theoretical point of view, there is a special value p_n of the price that corresponds to an *unbiased situation*: the situation where, on average, the willingness to pay is neutral, that is, there are as many agents likely to buy as not to buy. Since the expected willingness to pay of any agent i is $h + \theta_i + J/2 - p$, its average over the set of agents is $h + J/2 - p$. Thus, the neutral state is obtained for

$$p_n = h + J/2. \tag{1.26}$$

In the large N limit, even at finite T (a case not discussed in the present Chapter), symmetry breaking may occur: in spite of this neutrality in individual preferences, in the equilibrium state there is a majority of agents adopting the same behaviour (to buy or not to buy).

At $T = 0$ (deterministic dynamics), there is an interesting hysteresis phenomenon. The model has been shown to describe many physical systems (Sethna [66]), and has been applied to financial and economic systems (see [10,64], and Chapters **??** and **??**, this book).

Equilibrium for a given price. The simplest case with social effect is the "global" externality case with homogeneous interactions and full connectivity, i.e. with neighbourhood size $n = N - 1$, and consequently $J_\vartheta = J/(N-1)$. This is equivalent to the mean field theory in physics. In the McF case, the probability of having a positive payoff at price p is given by the distribution 1-$F(z_i)$, where F is the same logistic function as defined in equation (1.25). On the other hand, in the TP case, let us assume that the agents make repeated choices, and that the time varying components $\epsilon_i(t)$ are drawn at each time t from the *same* logistic distribution as the θ_i. In this special situation, the equilibrium distribution of choices for a given price is the same in both cases. In the McF case, it is convenient to identify a *marginal customer* indifferent between buying and not buying. Let $h_m = h + \theta_m$ be his private component of willingness to pay. This *marginal customer* has zero surplus ($V_m = 0$), that is:

$$\theta_m = p - h - \frac{J}{N-1} \sum_{k \in \vartheta} \omega_k \tag{1.27}$$

Consider the penetration rate η, defined as the fraction of agents that choose to buy, (i.e. $\theta_i > \theta_m$): $\eta = 1 - F(\theta_m)$. Then, in the large N limit,we have:

$$\theta_m \approx z(p) = p - h - \eta J. \tag{1.28}$$

This approximation of (1.27) allows us to define η as a fixed point. With the logistic distribution, we have:

$$\eta = 1 - F(z(p)) = 1/(1 + \exp(\beta z(p))) \tag{1.29}$$

Let us note that this fixed-point equation (1.29) is formally equivalent to the individual expectation for ω_i in the TP case (1.25).

The supply side. On the supply side, we consider a monopolist facing heterogeneous customers in a risky situation where the monopolist has perfect knowledge of the functional form of the agents surplus functions and the related maximisation behaviour (1.20). He also knows the statistical (logistic) distribution of the idiosyncratic part of the reservation prices (h_i). But, in the market process, the monopolist cannot observe any *individual* reservation price. Assume the simplest scenario of "global" externality, where the interactions are the same for all customers, as in equation (1.23). As just seen, in this case the TP model and the McF one have the same equilibrium states. Thus, hereafter we discuss only the McF model.

In this case, the social influence on each individual decision is equal to ηJ, where η (the fraction of customers) is observed by the monopolist. That is, for a given price, the expectation of the number of buyers is given by equation (1.29). Assuming null cost for simplicity, the monopolist can maximize his expected profit $p\,N\eta$. Since in this mean field case, this profit is proportional to the total number of customers, one is left with the following maximization problem:

$$p_M = \arg\max_p \Pi(p), \quad \text{with} \quad \Pi(p) \equiv p\eta(p), \tag{1.30}$$

where $\eta(p)$ is the solution to the implicit equation (1.29). Thus, p_M satisfies $d\Pi/dp = 0$, which gives $d\eta/dp = -\eta/p$ at $p = p_M$. Deriving the implicit equation (1.29) with respect to p, we obtain a second expression for $d\eta/dp$. Thus, at $p = p_M$:

$$-\frac{d\eta}{dp} = \frac{f(z)}{1 - Jf(z)} = \frac{\eta}{p}, \tag{1.31}$$

where z, defined in (1.28), has to be taken at $p = p_M$, and $f(z) = dF(z)/dz$ is the probability density. Because the monopolist observes the demand level η, we can use equation (1.29) to replace $F(z)$ by $1-\eta$. After some manipulations, equation (1.31) gives an inverse supply function $p^s(\eta)$, and equation (1.29) an inverse demand function $p^d(\eta)$:

$$p^s(\eta) = \frac{1}{\beta(1-\eta)} - J\eta \tag{1.32}$$

$$p^d(\eta) = h + J\eta + \frac{1}{\beta}\ln\frac{1-\eta}{\eta} \tag{1.33}$$

Finally, we obtain p_M and η_M at the intersection between supply and demand:

$$p_M = p^s(\eta_M) = p^d(\eta_M). \tag{1.34}$$

As might be expected, the result for the product βp_M depends only on the two parameters βh and βJ. Indeed, the variance of the idiosyncratic part of the reservation prices fixes the scale of the important parameters, and in particular of the optimal price.

Let us first discuss the case where $h > 0$. It is straightforward to check that in this case there is a single solution η_M. It is interesting to compare the value of p_M with the value p_n corresponding to the neutral situation on the demand side (1.26). For that, it is convenient to rewrite equation (1.33) as

$$\beta(p - p_n) = \beta J(\eta - 1/2) + \ln[\eta/(1-\eta)]. \tag{1.35}$$

This equation gives $p = p_n$ for $\eta = 0.5$, as it should. For this value of η, equation (1.32) gives $p = p_n$ only if $\beta(h + J) = 2$: for these values of J and h, the monopolist maximizes his profit when the buyers represent half of the population. When $\beta(h + J)$ increases above 2 (decreases below 2), the monopolist's optimal price decreases (increases) and the corresponding fraction of buyers increases (decreases). Finally, if there are no social effects ($J = 0$) the optimal price is a solution of the implicit equation:

$$p_M = \frac{1}{\beta F(p_M - h)} = \frac{1 + \exp\left(-\beta(p_M - h)\right)}{\beta}. \tag{1.36}$$

The value of βp_M lies between 1 and $1 + \exp(\beta h)$. Increasing β lowers the optimal price for the monopolist: since the variance of the distribution of willingness to pay gets smaller, the only way to keep a sufficient number of buyers is to lower the prices.

Consider now the case with $h < 0$, that is, on average the population is not willing to buy. Due to the randomness of the individual's reservation prices, $h_i = h + \theta_i$, the surplus may be positive but only for a small fraction of the population. Thus, we would expect that the monopolist will maximize his profit by adjusting the price to the preferences of this minority. However, this intuitive conclusion is not supported by the solution to equations (1.34) when the social influence represented by J is strong enough. The optimal monopolist's strategy shifts abruptly from a regime of high price and a small fraction of buyers to a regime of low price with a large fraction of buyers as βJ increases. This behaviour is analogous to what is called a first order phase transition in physics [67] (see Galam, Chap. **??** this book): the fraction of buyers jumps at a critical value of the control parameter βJ_1 from a low to a high value. Before the transition, above a value $\beta J_0 < \beta J_1$ the equations (1.34) present already several solutions. Two of them are local maxima of the monopolist's profit function, and one corresponds to a local minimum. The global maximum is the solution corrsponding to a high price with few buyers for $\beta J < \beta J_1$, and that of low price with many buyers for $\beta J > \beta J_1$. Figure 1.2 present these results for the particular value $\beta h = -2$, for which it can be shown analytically that $\beta J_0 = 4$, and $\beta J_1 \approx 4.17$ (determined numerically).

The preceding discussion only considers fully connected systems. The theoretical analysis of systems with finite connectivity is more involved, and requires numerical simulations. The simplest configuration is the one where each customer has only two neighbours, one on each side. The corresponding network is a ring, and has been analysed numerically by Phan et al. [64]), who show that the optimal monopolist's price increases both with the degree of the connectivity graph and the range of the interactions (in particular in the case of small worlds). Different sets of buyers' clusters may form, so that it is no longer possible to describe the externality with a single parameter, like in the mean field case. Further studies in cognitive economics are required in order to explore such situations (see also Phan, and Weisbuch and Stauffer, in this book for considerations about the dynamic aspects of these systems).

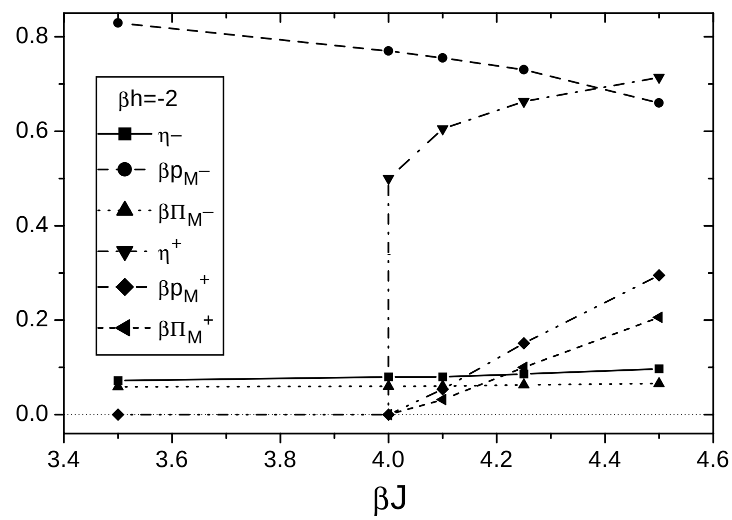

Fig. 1.2. Fraction of buyers η, optimal price βp_M and monopolist profit ηp_M, as a function of the social influence, for $\beta h = -2$. The upperscripts $-$ and $+$ refer to the two solutions of equations (1.34) that are relative maxima

1.4 Market organisation with search and price dispersion

In this section, we consider the modeling of the buyer's behaviour when faced with a multiple choice situation. Here we address the question of the choice function, quite often assumed to be a logit function . In the context of adaptive behaviour, Nadal et al. [60] show that this particular choice function can be justified as an *exploration-exploitation* compromise as we explain now.

We consider an agent in a context of an iterated game: at each period the agent makes use of previous experience to select a seller among N sellers. We limit the discussion to the case where agents only use past private information and have no access to public information, concerning for instance the behavior of other agents. Such a framework was studied in Weisbuch et al. [73] and Nadal et al. [60] for the modelling of the wholesale fish market in Marseille (see also [49]).

At each period of time the agent must choose one between N sellers; *strategy* j is to decide to visit (and make a transaction with) seller j. The buyer makes expectations $\hat{V}_j$ on the surplus he would get by visiting seller j (that is, he has an expectation on the result of strategy j). In the context of the fish market, this surplus is the buyer's profit (typically an owner of a restaurant who buys fish to be cooked and served in his restaurant). The optimal strategy for getting the best surplus at the next transaction is thus simply to choose the seller j that corresponds to the largest value $\hat{V}_j$.

In the context of bounded rationality, especially in a random and possibly non stationary environment, the agent cannot predict exactly the time evolution of V_j; he has to estimate it from past experience. Indeed the market can vary in time because of external events or because the sellers' strategies move.

In an adaptive rationality framework, each time the agent makes a transaction with seller j, he updates his estimate $\hat{V}_j$:

$$\hat{V}_j(t+1) = \Phi(\hat{V}_j(t), V_j(t)) \tag{1.37}$$

where $V_j(t)$ is the actual surplus obtained at time t, and $\Phi()$ is some appropriate learning rule. A simple and obvious rule is

$$\Phi(\hat{v}, v) = \lambda\hat{v} + (1-\lambda)v \tag{1.38}$$

where λ is some parameter between 0 and 1: with such a rule $\hat{V}_j(t)$ is a *moving average* estimate of V_j. Another and possibly more elaborated rule may be considered, in particular rules taking into account some rational expectation on the evolution of the sellers' strategies. Whenever there are no posted prices, or if the cost for comparing prices is too high (e.g. if the sellers are quite far apart from each other), the buyer can update his information on a given seller only by visiting him: the agent must then visit the different sellers and not only the one for which the expected surplus is maximal.

Hence, the buyer wants to find a compromise between maximizing his surplus at the next transaction, and keeping the best possible knowledge of the market in order to be able to make good choices in the future. This compromise will be sought in the form of a mixed strategy: we denote by q_j the probability of visiting seller j,

$$\sum_j q_j = 1. \tag{1.39}$$

Exploration requires him to visit every seller as frequently as possible. Optimal exploration would then correspond to the uniform distribution, $q_j = q_j^0 \equiv 1/N$. A proper measure of the similarity between this uniform distribution $\{q_j^0\}_{j=1}^N$ and the actual distribution $\boldsymbol{q} \equiv \{q_j\}_{j=1}^N$ is the entropy $\mathcal{S}$,

$$\mathcal{S}(\boldsymbol{q}) = -\sum_j q_j \ln q_j. \tag{1.40}$$

The entropy is a measure of uncertainty in the occurrence of the events $j = 1, ..., N$. In the context of information theory[6] (see [52] for a discussion of the relevance of information theory in theoretical economics), it is the minimal *amount of information* (measured in bits if the logarithm in (1.40) is taken in base 2) required in order to encode the set of events.

Given a mixed strategy, the mean expected surplus at time t is

$$E(\hat{V}(t)|\boldsymbol{q}) = \sum_j q_j \hat{V}_j(t) \tag{1.41}$$

The agent may then choose to maximize his expected utility under the constraint that the information on the market remains large enough. This can be written as

$$\boldsymbol{q} = \arg\max[E(\hat{V}(t)|\boldsymbol{q}) + T\mathcal{S}(\boldsymbol{q}) + G(\sum_j q_j - 1)] \tag{1.42}$$

where $\mathcal{S}$ is the entropy (1.40), and T and G are Lagrange multipliers associated with the information and the normalization constraints, respectively.

This optimization problem is formally equivalent to the standard *maximum entropy principle*, or "MaxEnt" principle, well known in statistical inference (see Jaynes [42]) and at the basis of the formal construction of statistical mechanics (see e.g. Balian [4], and Chap. **??** by Gordon, this book). To see that one can first make explicit the correspondence with the physics terminalogy: $E(\hat{V}(t)|\boldsymbol{q})$ plays the role of *minus* the mean energy, $\mathcal{E}(\boldsymbol{q}) = -E(\hat{V}(t)|\boldsymbol{q})$, of a system which has energy $-\hat{V}_j$ when in state j, and T plays the role of the temperature (at $T = 0$, the agent maximizes his surplus, the physical system minimizes its energy). Then one can write (1.42) as

$$\boldsymbol{q} = \arg\min[\mathcal{E}(\boldsymbol{q}) - T.\mathcal{S}(\boldsymbol{q}) - G.(\sum_j q_j - 1)] \tag{1.43}$$

The maximum entropy principle is the dual version of the exploration-exploitation compromise (1.43):

$$\boldsymbol{q} = \arg\max[\mathcal{S}(\boldsymbol{q}) - \beta\mathcal{E}(\boldsymbol{q}) + \gamma(\sum_j q_j - 1)] \tag{1.44}$$

with the correspondence $\beta = 1/T$. At the optimum, the right hand side of (1.44) is the opposite of (T times) the free energy.

It is easy to derive the optimal solution of (1.42) - or equivalently (1.44) -, which is precisely the logit rule:

$$q_j = \frac{1}{Z} \exp \beta \hat{V}_j(t) \tag{1.45}$$

with the normalization constant Z, the 'partition function', given by $Z = \sum_j \exp \beta \hat{V}_j(t)$.

The entropy is the only function of the q_j satisfying properties that makes it a reasonable quantitative measure of information [42]. In the present context, it can be seen as an appropriate cost for the search for information itself - given that the possible *monetary* costs, such as the cost of driving to the seller's location, are already taken into account in $\hat{V}_j$.

In the case of the wholesale fish market in Marseille, empirical data shows a bimodal distribution of buyers' behaviour: some buyers randomly choose the seller they will visit, and others have strong preferences, almost always visiting the same seller. Modelling with a logit choice function (or actually other choice functions with qualitative behaviour similar to the logit function), one can assume that each buyer has his own logit parameter β. As we have already seen in previous sections, phase transitions may occur as a function of β: buyers with β above some critical value β_c will almost always select the same seller, whereas buyers with β below the critical value will continue to explore all the sellers. Remarkably, this will occur even if all the profits are identical, $V_j(t) = V$ (see [49,60] for details).

To conclude, we have seen that the logit choice function can be viewed as resulting from the maximization of a cost function which expresses a compromise between exploration - keeping information about the market - and exploitation - making the largest surplus at the next transaction. This can be understood either as the result of the search for an optimal mixed strategy by the agent, or as the statistical description, by a seller, of the buyers' behaviour. Of course other approaches to the *exploration-exploitation* paradigm exist. In particular, deterministic strategies can be defined (see Gittins [35], Bourgine [11]). We note, however, that these approaches have been mainly studied in the context of a stationary environment. In the mixed strategy approach, stationarity is assumed only on a time scale on which the past information is taken into account (that is, for the simple model (1.38), the time scale defined by the parameter λ). It is thus even possible to adapt this time scale from the observations of the fluctuations of the observations themselves.

1.5 Conclusion

This Chapter reviews and extends some specific microeconomic models that take into account social influence in individual decisions, and doing so by taking advantage of a formalism inspired from statistical mechanics. Starting with a framework suggested by Durlauf, Blume and Brock (DBB), we introduced three classes of models shifting progressively from rational towards adaptive expectations. The first two are based on a generalized version of the standard economic model of discrete choices that include the social influence as an additive argument. In the DBB framework, the relevant concepts are those of game theoretic and rational expectations. The second class of models deals with a discrete choice monopoly market with demand externality. On the demand side, customers are assumsed to be myopic and non strategic. On the supply side, the monopolist is the only cognitive agent in this process. The last model analyzed is devoted to the market organization resulting from a search process with adaptive expectations over heterogeneous sellers, based on the model of the Marseille wholesale fish market. In all these models, we mainly considered the simplest case of complete connectivity, which corresponds to the so-called mean field theory in Physics.

Issues not usually addressed in the literature have also been discussed. In particular we compared models with deterministic and random idiosyncratic (heterogeneous) preferences. They correspond, to models with quenched and annealed disorder in statistical physics respectively. Interestingly, to comply with the most frequent assumptions in economics, we were led to consider situations that are unusual in physics, like having a logit distribution of the quenched variables.

We tried as far as possible to clarifiy the links with statistical mechanical models, with the aim of opening a discussion about the relevance of the methods and concepts of statistical mechanics in microeconomics. This is why we focused on a very restricted class of models. A review of the huge and ever growing literature of models of social influence in economics is clearly beyond the scope of this Chapter. A specific monography would certainly deserve tp be writter. However, we hope that this Chapter will provide useful insight into models that go beyond the usual rational agent approach, by taking explicitly into account the social effects and interactions between the individuals beliefs. This perspective raises numerous questions for cognitive economics. Some of them are more specifically considered in other Chapters of this book (Orléan, Baron et al., Galam, Weisbuch et al., etc.).

References

1. Allen B. (1982a) "A Stochastic Interactive Model for the Diffusion of Information", Journal of Mathematical Sociology, 8 p.265-281
2. Allen B. (1982b) "Some Stochastic Processes of Interdependent Demand and Technological Diffusion of an Innovation Exhibiting Externalities Among Adopters", International Economic Review 23 p. 595-608.
3. Anderson S.P., DePalma A, Thisse J.-F. (1992) *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge MA.
4. Balian R. (1992) *From Microphysics to Macrophysics*, Springer.
5. Bernheim B.D. (1994) "A Theory of Conformity". Journal of Political Economy, vol.102/5, 841-877.
6. Blahut R. E. (1988) *Principles and Practice of Information Theory*, Addison-Wesley, Cambridge MA.
7. Blume L.E. (1993) "The Statistical Mechanics of Strategic Interaction". Games and Economic Behavior, 5, 387-424.
8. Blume L.E. (1995) "The Statistical Mechanics of Best-Response Strategy Revisions". Games and Economic Behavior, 11, p.111-145.
9. Blume L.E., Durlauf S.N. (2001) "The Interaction-Based Approach to Socioeconomic Behavior", in Durlauf, Young (2001) eds., p.15-44.
10. Bouchaud J-Ph. (2000) "Power-laws in Economics and Finance: some ideas from physics", Quantitative finance 1, 105.
11. Bourgine P. (1998), "The compromise between exploration and exploitation: from decision theory to game theory", in J Lesourne and A Orléan Eds., op. cit., pp 397-416.

12. Brock W.A. (1993) "Pathways to Randomness in the Economy: Emergent Non-Linerarity and Chaos in Economic and Finance", Estudios Economicos 8(1) p.3-55. (SSRI Reprint 410 University of Wisconsin)
13. Brock W.A., Durlauf S.N.(2001) "Interaction Based Models", in: Heckman, Leamer (eds.) *Handbook of Econometrics* Volume 5, Ch 54, pp. 3297-3380, Elsevier Science B.V, Amsterdam.
14. Cont R. (1999) "Modeling economic randomness: statistical physics and market phenomena", in *Statistical Physics on the Eve of the 21st century: the James B. McGuire Festschrift*, pp 47 - 64, Singapore: World Scientific. (http://www.cmap.polytechnique.fr/ rama/papers/festschrift.ps)
15. Cont R., Bouchaud J-Ph. (2000) "Herd behavior and aggregate fluctuations in financial markets", Macroeconomic dynamics, Vol 4, 170 - 196.
16. Cohendet, Llerena, Stahn, Umbauer eds. (1998) *The Economics of Networks, Interactions and Behaviours*, Springer, Berlin.
17. Bulow J.I., Geanalopolos J.D., Klemperer P.D. (1985) "Multimarket oligopoly: strategic substitutes and complements", Journal of Political Economy, 93/3 pp.488-511.
18. Cooper R., John A. (1988) "Coordinating coordination failures in keynesian models", Quarterly Journal of Economics, 103 p. 441-463.
19. Dalle J.M. (1998) "Heterogeneity, interactions structures and the diffusion of technological innovation", in Lesourne and Orléan Eds., op.cit., pp.240-262.
20. Durlauf S.N. (1991) "Multiple equilibria and persistence in Aggregate Fluctuations", American Economic Review, 81, p. 70-74.
21. Durlauf S.N. (1993) "Nonergodic economic growth", Review of Economic Studies, vol. 60 no 203, p. 349-366.
22. Durlauf S.N. (1994) "Path Dependance in Aggegate Output", Industrial and Corporate Change, 1, p.149-172.
23. Durlauf S.N. (1996a) "A Theory of Persistent Income Inequality", Journal of Economic Growth, 1 p.349-366.
24. Durlauf S.N. (1997) "Statistical Mechanics Approaches to Socioeconomic Behavior", in Arthur, Durlauf, Lane (eds). (1997) op.cit.
25. Durlauf S.N. (1999) "How can Statistical Mechanics contribute to social science?", Proceedings of the National Academy of Sciences, Vol.96, pp.10582-10584.
26. Durlauf S.N. (2001) "A Framework for the Study of Individual Behavior and Social Interactions", Working paper, University of Wisconsin-Madison, May.
27. Durlauf S.N., Young P. (2001) eds., *Social dynamics*, The MIT Press, Cambridge Ma.
28. Econophysics Forum, http://www.unifr.ch/econophysics/
29. Föllmer H. (1974) "Random Economies with many Interactig Agents", Journal of Mathematical Economics 1, p.51-62.
30. Galam S., (1982) "A New Multicritical Point in Anisotropic Magnets. III. Ferromagnets in both a Random and a Uniform Longitudinal Field", Journal of Physics C, 15, pp. 529-545.
31. Galam S. (1986) "Majority rule, hierarchical structure and democratic totalitarism", Journal of Mathematical Psychology, 30, p.426.
32. Galam S., AharonyA. (1980) "New multicritical points in anisotropic magnets, I. Ferromagnets in a random longitudinal field", J. Phys. C: Solid State Phys. 13, 1065-1081.

33. S. Galam and A. Aharony (1981) "A new multicritical point in anisotropic magnets, II. Ferromagnet in a random skew field", J. Phys. C: Solid State Phys. 14, 3603-3619.
34. Galam S., Gefen Y., Shapir Y. (1982) "Sociophysics: A Mean Behavior Model for the Process of Strike", Mathematical Journal of Sociology, 9, pp. 1-13.
35. Gittins J C (1989) *Multi-armed bandit: allocation indices*, John Wiley & Sons
36. Glaeser E., Bruce I. Sacerdote I., Scheinkman J.A. (1996) "Crime and Social Interactions", Quarterly Journal of Economics, CXI p.507-548
37. Glaeser E., Kallai H.D., Scheinkman J.A., Shleifer A. (1992) "Growth in Cities", Journal of Political Economy, 100-6 p.1126-1152.
38. Heymann D., Sanguinetti P. (1998) "Business cycles from misperceived trends", Economic Notes by Banca Monte dei Paschi di Siena SpA, vol. 27, no. 2-1998, 205-232.
39. Hildenbrand W. (1971) "Random Preferences and Equilibrium Analysis", Journal of Economic Theory, 3 p.414-429.
40. Ioannides Y.M. (1990) "Trading Uncertainty and Market Structure", International Economic Review 31(3) p.619-638.
41. Ioannides Y.M. (1997) "Evolution of Trading Structures", in Arthur, Durlauf, Lane (1997) eds., op. cit. p.129-161.
42. Jaynes E.T., *Probability Theory: The Logic of Science*, http://omega.math.albany.edu:8008/JaynesBook.html; Jaynes E.T. (1982) "On the rationale of maximum-entropy methods", Proceedings of the IEEE, vol. 70, no. 9, pp. 939-952.
43. Kindermann R., Snell J.L. (1980a) *Markov Random Fields and their Applications*, American Mathematical Society, Providence, Rhode Island
44. Kindermann R., Snell J.L. (1980b) "On the relation between markov random fields and social networks", Journal of Mathematical Sociology, 7, p.1-13.
45. Kirman A.P. (1983) "Communications in Markets: A suggested Approach", Economic letters, 12 p. 1-5.
46. Kirman A..P. (1997a), "The Economy as an Interactive System", in Arthur B., Durlauf S., Lane D. (eds.), *The Economy as an Evolving Complex System II*, Santa Fe Institute Studies in the Sciences of Complexity, Volume XXVII, Addison-Wesley, pp. 491-531.
47. Kirman A.P. (1997b), "The Economy as an Evolving Network", Journal of Evolutionary Economics, 7, pp. 339-353.
48. Kirman A.P. (1998) "Economies with interacting agents", in Cohendet et al., op.cit., p. 17-52.
49. Kirman A.P. (2001), "Market Organization and Individual Behavior: Evidence from Fish Markets", in Rauch J.E., Casella A. (eds.), *Networks and Markets*, Russell Sage Pub, p. 155-195.
50. Kirman A..P. (2003), "Economic Networks", in Bornholdt S., Schuster H.G., eds. (2003) *Handbook of Graphs and Networks, from the Genome to the Internet*, Whiley-VCH Weinheim, p. 273-294.
51. Kirman A., Oddou C., Weber S. (1986) "Stochastic communication and coalition formation", Econometrica, 54 p. 129-138.
52. Lancry P. J. (1982) *Thorie de l'information et conomie*, Economica
53. Lesourne J., Orléan A. (1998) eds. *Advances in Self-Organization and Evolutionay Economics*, Economica, Londres Paris.

54. Levy M., Levy H., Solomon S. (2000) *Microscopic simulation of financial markets: from investor behavior to market phenomena*, Academic Press, San Diego, CA
55. Mansky C. (1993a) "Dynamic choice in social settings. Learning from the experience of others", Journal of Econometrics, 58 p.121-136
56. Mansky C. (1993b) "Identification of Endogenous Social Effects: The Reflexion Problem", Review of Economic Studies 60 (July): 531-542.
57. Mantegna R.N., Stanley H.E. (2000) *An introduction to Econophysics, Correlations and complexity in Finance*, Cambridge University Press.
58. McFadden D. (1974) "Econometric Analysis of Qualitative reponse Models", in Griliches Intrilligator (eds.), *Handbook of Econometrics* Vol II Elsevier Science B.V, Amsterdam.
59. Nadal J.P., Phan D., Gordon M. B. and Vannimenus J. (2003) "Monopoly Market with Externality: An Analysis with Statistical Physics and ACE", 8th Annual Workshop on Economics with Heterogeneous Interacting Agents (WEHIA), Kiel, May 29-31 (http://halfrunt.bwl.uni-kiel.de/vwlinstitute/gwrp/wehia/papers/nadal.pdf)
60. Nadal J.P., Weisbuch G., Chenevez O., Kirman A. (1998) "A Formal Approach to Market Organisation: Choice Functions, Mean Field Approximation and Maximum Entropy Principle", in Lesourne, Orléan eds., *Advances in Self-Organization and Evolutionary Economics*, Economica, London, pp. 149-159
61. Onsager, L.(1944) "A 2D model with an order-disorder transition", Physical Review, 65:117-149.
62. Orléan A. (1995) Bayesian Interactions and Collective Dynamics of Opinion: Herd Behaviour and Mimetic Contagion. Journal of Economic Behavior and Organization, 28, pp. 257-74.
63. Lesourne J. and Orléan A., eds. (1998) *Advances in Self-Organization and Evolutionary Economic*, Economica.
64. Phan D., Pajot S., Nadal J.P. (2003) "The Monopolist's Market with Discrete Choices and Network Externality Revisited: Small-Worlds, Phase Transition and Avalanches in an ACE Framework", Ninth annual meeting of the Society of Computational Economics, University of Washington, Seattle, USA, July 11 - 13, 2003.
65. Schelling T.S. (1973) "Hockey helmet, Concealed Weapons and Daylight Saving: A Study of Binary Choices with Externalities", Journal of Conflict Resolution, 17, p. 381-48.
66. Sethna J.P., Dahmen K., Kartha S., Krumhansl J.A., Roberts B.W., Shore J.D. (1993), "Hysteresis and Hierarchies: Dynamics of Disorder-Driven First-Order Phase Transformations", Physical Review Letters, 70, pp. 3347-3350.
67. Stanley H.E. (1971) *Introduction to phase transitions and critical phenomena*, Oxford University Press.
68. Thurstone L.L. (1927) "Psychological Analysis", American Journal of Psychology, 38, pp.368-398.
69. Gordon M.B., Nadal J.P., Vannimenus J., Phan D. (2003) Work in progress.
70. Wasserman, S. Faust, K. (1994) *Social Network Analysis: Methods and Applications*, Cambridge University Press.
71. Weidlich W. (1971) "The use of stochastic models in sociology", Collective Phenomena, 1, p. 51-59
72. Weidlich W. (1972) "The statistical description of polarization phenomena in society", J. Math. Stat. Psychol., 24, p. 251-266.

73. Weisbuch G., Kirman, A. Herreiner D. (1998) "Market Organisation", in Lesourne, Orléan (1998) eds., op.cit.