Small Worlds and Phase Transition in Agent Based Models with Binary Choices.

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KEYWORDS

Agent based computational economics, interaction network structures, critical threshold, symmetry breaking and phase transition, small world, evolutionary games, monopoly market with discrete choice and local externality.

ABSTRACT

This paper reviews some aspects of complex collective behaviour in an agent-based framework. It examines the effects of interaction structures (lattices, small worlds) on dynamic properties of simple binary choices models. When individual actions become interdependent, complex dynamics may arise. In order to illustrate such a phenomenon, a very simple evolutionary game is presented, using a spatial prisoner dilemma framework. The effect upon the global demand of local externalities between consumer is then investigated, in the case of discrete choice. The role of symmetry breaking (phase transition) in the diffusion process is underlined as well as the incidence of network structure on the optimal price for a monopolist in the simplest case, where this stationary optimal equilibrium is unique.

INTRODUCTION / ABSTRACT

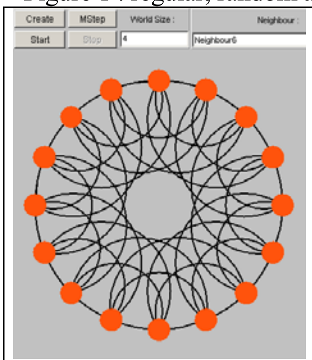
In this paper, we explore the effects of *interaction structures* upon dynamical properties of two models of agent based binary choices. Simulations are supported by Moduleco (Phan, Beugnard, 2001; Phan, 2003). A very interesting feature of complex adaptive systems dynamics (Weisbuch, 1991, Schuster 2001) is a *classical* phenomenon in the physics of disordered systems: *phase transition*. In the simplest case of phase transition, the system only bifurcates between two opposite states, but many other dynamic behaviours may arise. Physicists attribute such phenomena to *symmetry breaking* (Anderson and Stein, 1983). Broken symmetry gives rise to the appearance of a new phenomenon that did not exist in the symmetric phase. When individual actions are made to be interdependent, complex dynamics may arise. That is the case, for instance, when agents locally interact over a specific network. In such cases, Axtell (2000) has underlined the effects of distinct agent interaction structures in multi-agent models. In this paper, we review the effect of various network structures. Regular network (lattices) and random networks represent two limiting cases, the so-called "small-world" networks (Watts and Stogatz, 1998; Watts, 1999) being an intermediate form between these two extremes. In order to *illustrate* such a phenomenon, a very simple evolutionary game with a spatial prisoner dilemma framework is presented in the first section. The second section is devoted to the effect of the network structure, supporting a local externality between consumer preferences upon the global demand, in the simple case of binary choice. The role of symmetry breaking (phase transition) in the diffusion process is underlined as well as the incidence of network structure upon the optimal price for a monopolist

THE ROLE OF THE TOPOLOGY OF INTERACTION STRUCTURES IN COLLECTIVE DYNAMICS.

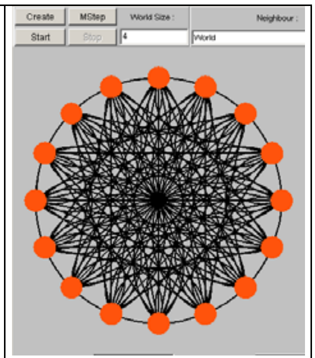
The so-called "small world "

Following an important body of literature in the field of socio-psychology and sociometrics, initiated by Milgram, (1967), the "six degrees of separation" paradigm of a "small world", Watts and Strogatz (1998) proposed a suggested formalisation in the field of disordered systems.

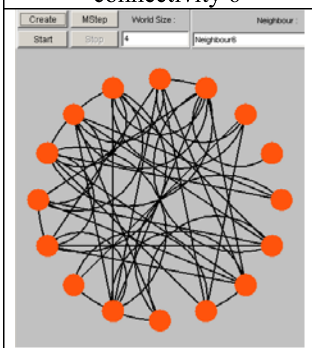
Figure 1 : regular, random and "small world" networks



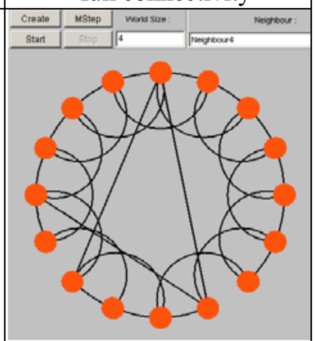
1.1. Regular network
- connectivity 6



1.2.- Regular network
- full connectivity



1.3. - Random network
- connectivity constrained



1.4. - Small World
Connectivity 4. 3 links rewired

Source: Phan, (2003)

The original Watts and Strogatz (WS) "small world" starts from a regular network where n agents are on a circle (dimension one, periodic lattice) and each agent is linked with his 2.k closest neighbours. In the WS rewiring algorithm, links can be broken and randomly rewired with a probability p. In this way, the mean connectivity remains constant, but the dispersion of the existing connectivity increases. For p = 0 we have a regular network and for p =1 a random network. Intermediate values between 0 and 1 correspond to the mixed case, where a lower p corresponds to a more local neighbour-dependent network. In Moduleco, the actual algorithm took h nodes, broke i links for each of these nodes and randomly rewired the broken links with other nodes. We have a parameter q = h.i/n which plays a similar role to p.

A large range of small world properties is now well known (Watts 1999, Newman 2000). Barthelemy et al. (2000) provide a typology of small world, with related properties, including both Watts-Strogatz and some varieties of "scale free" topologies. In Economics, the small world has been applied to bilateral games (Jonard et al. 2000, Jonard 2002), the knowledge diffusion process and innovation (Cowan and Jonard, 2001 ; Cowan et ali. 2002) and market organisation (Wilhite, 2001; Vannimenus et al. 2003; Nadal and Phan 2003)

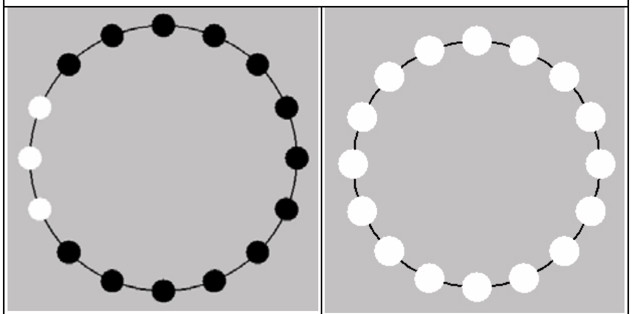
Generic evolutionary model of the prisoner dilemma : the simplest case

In the generic model, agents play a symmetric game (here, a prisoner dilemma) with each of their neighbours on a lattice. In such a game, defection is the only Nash equilibrium of the one- shot game. Axelrod (1984) among other, have explored forward looking repeated games. In this perspective, bilateral games plus a convenient revision rule constitute a special kind of *evolutionary game*. In such games, complex dynamics may arise, making the simulation very useful. In our game, at a given period of time, each agent plays the same strategy (S1 : co-operation or S2 : defection) in all these bilateral games. At the end of the period, each agent observes the strategy of his neighbours and the cumulated payoff of their strategy. But the agent has no information at all about the other games played by his neighbours. He observes only the cumulated payoff linked with this strategy. At each period of time, agents update their strategy, given the payoff of their neighbours. Assuming myopic behaviour, the simplest rule is to adopt the strategy of the last neighbourhood best (cumulated) payoff. Another rule used by Jonard et al. (2000) and Jonard (2002) is to adopt the strategy of the last neighbourhood best *average* (cumulated) payoff. This latter rule is less mimetic, because one may interpret this revision rule as a kind of estimator of the expected cumulated payoff of a given strategy (for the model maker, that is a conditional expected payoff given the strategies of the neighbour's neighbourhood).

In the simplest model (Figure 2), N agents play the symmetric game (prisoner dilemma) with each of their two neighbours on a circle (periodic, one-dimensional lattice). The revision rule is *the last neighbourhood best cumulated payoff*. If the payoff of the co-operation against themselves is sufficiently high (S1 against S1 > 91), defection (S2) is contained in a "*frozen zone*" of 3 agents. In other cases (S1 against S1 < 92), the whole population turns to defection. For N ≥ (3x3) this result is independent of the number of agents..

Figure 2 : co-operation and contained defection in the simplest one dimensional spatial game (Source: Phan, 2003)

(J_1,J_2)	J_1/S1	J_1/S2
J_2/S1	(X,X)	(176, 0)
J_2/S2	(0, 176)	(6, 6)
S1 : co-operation (black) ; S2 : defection (white)		



176 > X ≥ 92 : defection is contained in a "*frozen zone*"

6 < X ≤ 91 , the whole population turns to defection

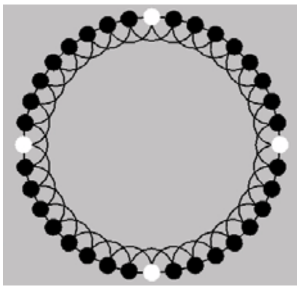
This simplest version (on a one dimensional periodic lattice) exhibits only a phase transition between two symmetric states: complete defection and quasi-complete co-operation (with contained frozen zone). More complex behaviour may arise when the connectivity increases, like in the May, Nowak (1992, 1993) model, where agents interact on a two dimensional periodic lattice (torus), or when the network is not a regular one, as in the next section. The introduction of random noise may also produce different results, as in the pioneering work of Ellison, 1992, in a larger class of symmetric games.

Introducing small world structure in a 4 neighbour regular lattice

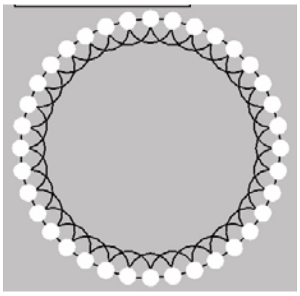
The following example is drawn from work in progress by Pajot, Phan to illustrate the power of rewiring in changing interactive environment by shifting the symmetry breaking point. For the spatial prisoner dilemma game (and a larger class of bilateral games), Jonard et al. (2000) have established (for the *best average payoff* rule) that the stability of co-operative coalitions depends on the degree of regularity in the structure of the network. In the following example, co-operation is unsustainable within a regular network, but becomes sustainable within a rewiring disorder. The core of the model is the same as that of the spatial prisoner dilemma, but with a one dimensional - periodic neighbour 4 structure (on a circle). To be clear, we have limited the population to N= 36 agents (32 co-operators for 4 defectors). According to the best neighbourhood payoff rule, each agent chooses the best cumulated payoff strategy in the neighbourhood as strategy revision rule. The aim of this exercise is to improve the strength of a network against *accidental* defection. That is, four *temporary* defectors are symmetrically introduced into the network. When the network is regular, defection is the winning strategy, and diffuses to the whole population. With the numerical values of defection payoffs in this game (Figure 3), this is the case for all payoffs of the co-operation against itself between 7 to 175 (that is, in the prisoner dilemma case).

Fig. 3 – Symmetric introduction of defection in a regular network of co-operators

(J_1,J_2)	J_1/S1	J_1/S2
J_2/S1	(170,170)	(176, 0)
J_2/S2	(0, 176)	(6, 6)
S1 : co-operation (black) ; S2 : defection (white)		



t = 0

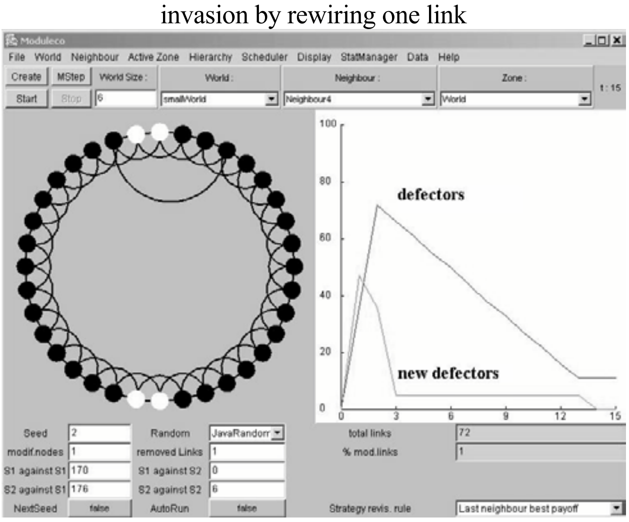


equilibrium

Within the *regular network* case, the number of defectors grew and became stable for 100% of the population

Source: Pajot, Phan, (2003)

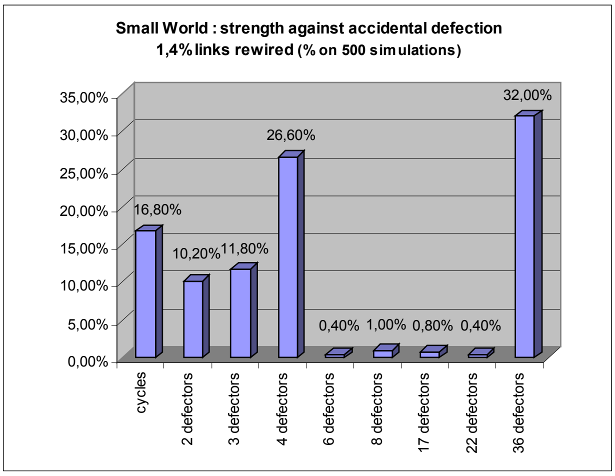
Fig. 4 –Making the network robust against defectors' invasion by rewiring one link



In some cases, changes in the structure of the networks by minor modifications in the neighbourhood of some agents allow co-operation to protected against defection. The number of defectors increases at first and reaches roughly 60% of the population, but a rewired link may reverse this evolution in a second step. In such a case (figure 3), defection decreases towards stabilisation at 11 %.

Even if co-operation failed to be held in all case of the regular network, a one link rewiring is sufficient to limit to only 1/3 the percentage of cases with a totality or a majority of defectors. Moreover, in roughly one half of the cases, defectors are limited to four or less (as in figure 4). First results of simulations suggest that the percentage of stable co-operators becomes higher with sufficiently long range links, i.e. linked agents with a sufficiently distant local neighbourhood. The best averaging result (the weakest percentage of defection) occurs for 1 node modified with 2 rewired links.

Fig. 5 – Statistical results for 500 simulations



Source: Phan, (2003)

THE NETWORK STRUCTURE OF THE GLOBAL DEMAND WITH EXTERNALITY

This model deals with the simplest discrete choice problem (Anderson et ali. 1992): the binary choice. The agent buys one unit of given goods ($\omega_i = 1$) or does not buy it ($\omega_i = 0$). The model and results follow Vannimenus et ali. (2003) and Phan, Pajot, Nadal (2003).

A discrete choice model with externality: critical mass around the transition phase.

Agents have a linear willingness to pay. The surplus function $V_i(\omega_i)$ is positive if the agent buys a single unit of these goods and is null otherwise.

$$(1) \qquad \max_{\omega_i \in \{0,1\}} V_i(\omega_i) = \omega_i (h_i + \sum_{k \in \vartheta} J_{i\vartheta}.\omega_k - p)$$

The private component h_i is strictly deterministic, with a common sub-component h, and an idiosyncratic component γ_i:

$$(2) \qquad \frac{1}{N}\sum_N h_i = h + \frac{1}{N}\sum_N \gamma_i = h$$

Agents are randomly distributed on the network (random field) according to a parametric cumulative distribution

$F(\gamma)$ with mean = 0. , (Logistic distribution with parameter β, i.e. standard deviation $\sigma = \pi/(\beta.\sqrt{3})$.

Assuming *homogeneous local interactions* in each neighbourhood, we have symmetric $J_{i\vartheta} = J/N_{i\vartheta}$ for all influence parameters, where $N_{i\vartheta}$ is the number of neighbours around agent i and J a positive parameter. For a given neighbour k taking in the neighbourhood ($k\in \vartheta$), the social influence is $J_{i\vartheta}$ if the neighbour is a customer ($\omega_k = 1$), and zero otherwise. Individual influence is inversely proportional to the size of the neighbourhood. As the cumulated social effect is the sum of individual effects over the neighbourhood, social influence depends on the *proportion* of customers in the neighbourhood. In a *regular* network ($N_{i\vartheta}$ constant and equal to N_ϑ for all i) all individual effects have the same magnitude over the network (equal to $J_\vartheta = J/N_\vartheta$). Conversely, in a *small world* network or in a *random* network, the magnitude is inversely proportional to the size $N_{i\vartheta}$ of a given neighbourhood.

Following Gordon et al. (2003), it is possible to underline a formal equivalence between this economic model and *the Random Field Ising Model* (RFIM), at zero temperature of the statistical physics. Moreover, for physicists, this model with fixed heterogeneity belongs to the class of *quenched disorder* models, because each agent is subject to a non-uniform *external field*: h_i - p. From the economist's or the sociologist's point of view, customer behaviour depends on an individual threshold subject to social influence (Granovetter, 1978). Individual threshold depends on the value of surplus function (1), which is supposed to be maximised at the micro level. Economist's have to compare willingness to pay and the price. The former include both idiosyncratic component: ($\omega_i.h_i$), and social component (the so called *local field* for the physicists):

$$\omega_i . \sum_{k\in\vartheta} J_{i\vartheta}.\omega_k$$

Let us denote j_ϑ the (marginal) social influence of an agent in the neighbourhood $k\in\vartheta$ on i. Following Durlauff (1997), the assumption of $J_{i\vartheta} > 0$ means a positive influence of k upon i, i.e. a *strategic complementarity* (Bulow, Geanakoplos, Klemperer, 1985) between the choice of i & k.

That is, for a given variation in price, it is possible to observe the resulting variation in demand. Such variation could be induced by two different channels. On the one hand, a change in price may *directly* induce an individual adoption without any change in his neighbourhood. On the other hand, the change in price may induce one or more adoption(s) in the neighbourhood of an agent, and therefore, through social influence, the change in the consumer willingness to pay which may lead this agent to adopt indirectly. Such an indirect diffusion mechanism is known in literature as the domino effect, the bandwagon effect or avalanches. The most spectacular result in avalanches may be observed when all agents update their choices simultaneously (Fig 6)

Fig 6 : straight phase transition under "world" activation regime

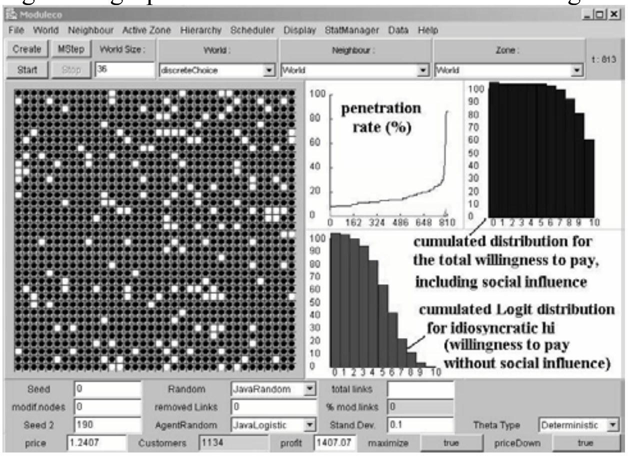


Figure 7 shows the chronology and the size of induced adoptions in the avalanche at the phase transition when p decreases from p = 1.2408 to p = 1.2407. This drives the system from an adoption rate of 20% towards an adoption rate of roughly 80%. The evolution follows a smooth path, with a first period of 19 steps, where the initial change of one customers leads to growing induced effects from size 2 to size 81 (6,25% of the whole population). After this maximum, induced changes decrease in 13 steps, including 5 of size one only.

Figure 7 : Chronology and sizes of induced adoptions in the avalanche when decrease from 1.2408 to 1.2407 (seed : 190)

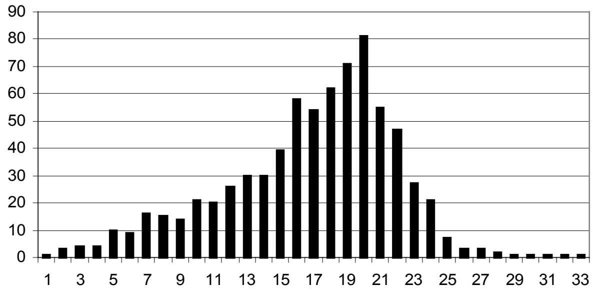
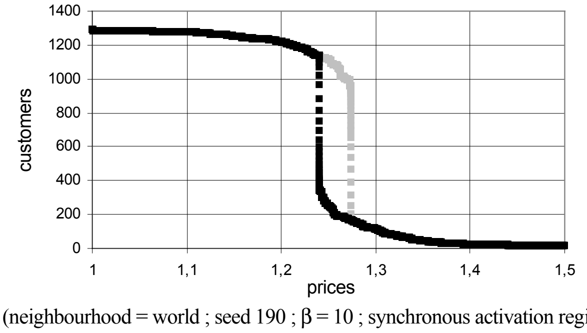


Figure 8 shows the set of the set of equilibrium positions for the whole demand system over all prices, incremented in steps of 10^{-4}, within the interval [1,20 - 1,30]. At the phase transition, one observes the so called hysteresis phenomenon around the theoretical *point of symmetry breaking*:

$$p^s = h + j/2 = 1,25$$

Figure 8 : hysteresis in the trade-off between price and customers

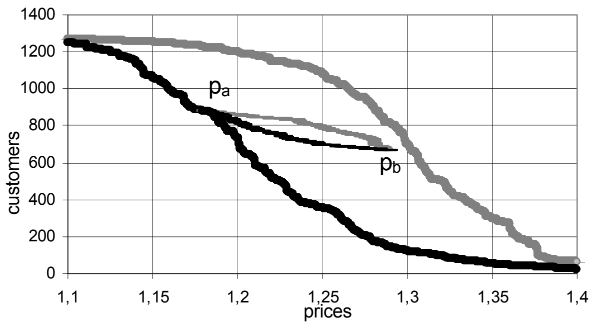


(neighbourhood = world ; seed 190 ; $\beta = 10$; synchronous activation regime)

Along the upstream equilibrium trajectory (with decreasing prices), a dramatic avalanche occurs when p decreases from 1.2408 to 1.2407, as may be seen in Fig 7. Along the downstream trajectory (with increasing prices) the externality effect induces a strong resistance of the demand system against a decrease in the number of customers. The phase transition threshold is here around p = 1,2745. At this threshold, the equilibrium adoption rate decreases dramatically from 73% to 12,9%.

The scope of such phenomenon depends both upon connectivity (i.g. the size of the neighbourhood) and upon the variance of idiosyncratic characteristics within the population. On the one hand, the size of avalanches decrease with the connectivity (Figure 9). On the other hand, the hysteresis interval (i.e. the price interval where there are two different equilibria for the same price) decreases when the variance of idiosyncratic characteristics increases. For example, given $\sigma \sim 1/\beta$, if β goes from 10 to 5, doubling σ, there is not any hysteresis at all).

Figure 9 : outer and inner hysteresis loop



(neighbourhood = 8 ; seed 190 ; β = 10 ; synchronous activation regime)

Finally, because this model is formally equivalent to the RFIM, it is not surprising to observe the same dynamics as with the later (Gordon et al., 2003). In particular, it is possible to observe the *return point memory effect* (Sethna et al., 1993). In Figure 9, the inner loop within the hysteresis outer loop illustrate such phenomenon. Starting from p_a when the price is growing to p_b, the demand decrease following the grey line. If the price return to p_a, the demand increase following the darker (black) line, drawing an inner hysteresis, while demand return exactly at the same level from which it left the outer loop. Such a property is very interesting for the so *called exploration-exploitation* process. For instance, a seller may explore some prices around their current position in order to learn the characteristics of the demand curve. At the present time, such a result, discovered by simulation have not general analytical solution.

Optimal monopoly pricing: the stationary case and the need for individual learning

On the supply side, Phan, Pajot, Nadal (2003) consider a monopoly faced with heterogeneous customers whose individual reservation prices are initially unknown, but for whom eq. (1) and all relevant parameters of the statistical distribution for the idiosyncratic component are well known. The simplest case with social effect is the « global » externality case with *homogeneous interaction*. In this case, the results are independent of the network structure. The adoption rate $n_c = N_c/N$ is:

$$(3) \qquad n_c = \frac{N_c}{N} = \int_{\gamma_m}^{\infty} dF(x) = 1 - F(\gamma_m)$$

where $h_m = h + \gamma_m$ is the private component of the willingness to pay of the marginal customer indifferent between buying and not buying (were J_ϑ denote the same social influence parameter for all agents)

$$(4) \qquad \gamma_m = p - h - \sum_{k \in \vartheta} J_\vartheta . \omega_k = p - h - N_c . J_\vartheta$$

Monopolist maximises his expected profit (assuming null cost):

$$(5) \qquad \max_p \Pi(p) = p.\left(1 - F\left(p - h - N_c . J_\vartheta\right)\right).N$$

With social effect ($j_\vartheta \neq 0$), the optimal price is a fixed point of the following equation (9):

$$(9) \qquad p^* = H . \frac{1 - F\left(p^* - u_1 - N_c(p).J_\vartheta\right)}{f\left(p^* - u_1 - N_c(p).J_\vartheta\right)}$$

$$\text{with:} \quad H = \left(1 - f\left(p^* - u_1 - N_c(p).J_\vartheta\right).J_\vartheta\right)$$

Hence, without social effect we have: $p^* = \dfrac{1 - F(p-h)}{f(p-h)}$

Table 1. The distribution of optimal equilibrium pricing.

1296 Agents	optimal prices	adoptors	profit	Adoption rate	q
no externality	**0,8087**	**1135**	**917,91**	***87,58%***	
Neighbour2	**1,0259**	**1239**	**1 271,17**	**95,60%**	
Neighbour 4	**1,0602**	**1254**	**1 329,06**	**96,76%**	
Neighbour 4_130x2	1,0725	1250	1 340,10	96,45%	5%
Neighbour 4_260x2	1,0810	1244	1 344,66	95,99%	10%
Neighbour 4_520x2	1,0935	1243	1 358,86	95,91%	20%
Neighbour 4_1296x2	1,1017	1237	1 362,35	95,45%	50%
Neighbour 6	**1,0836**	**1257**	**1 361,48**	**96,99%**	
Neighbour 6_260x2	1,0997	1252	1 376,78	96,60%	7%
Neighbour 6_520x2	1,1144	1247	1 389,05	96,22%	13%
Neighbour 6_1296x2	1,1308	1241	1 403,03	95,76%	33%
Neighbour 6_1296x4	1,1319	1240	1 403,02	95,68%	66%
Neighbour 8	**1,1009**	**1255**	**1 381,89**	**96,84%**	
Neighbour 8 260 x 2	1,1169	1249	1 395,43	96,37%	5%
Neighbour 8 520 x 2	1,1306	1245	1 407,20	96,06%	10%
Neighbour 8 1296x2	1,1461	1238	1 419,28	95,52%	25%
Neighbour 8 1296x4	1,1474	1239	1 421,97	95,60%	50%
Neighbour 8 1296x6	1,1498	1238	1 423,84	95,52%	75%
world	**1,1952**	**1224**	**1 462,79**	**94,44%**	

In the simplest case, when the optimal equilibrium solution is outside the hysteresis, and the stable equilibrium is unique (Phan, Pajot, Nadal 2003). Equations (9) allow to calculate the exact solution in this two extreme cases.

The mean result of 100 simulations with a finite population and decreasing prices (Table 1) provides coherent results. Simulations also provide numerical value for local networks (both lattices and small world), while any analytic solution is now available. With scale constrained small world, equilibrium price grows with the connectivity and with the degree of randomness in the network (i.e. The percentage q of link rewired). It should be noted that, in case of local interactions, upstream equilibrium (decreasing prices) may be different of upstream one (hysteresis with two stable equilibria of a single price), because the hysteresis interval increase when the connectivity decrease.

As conclusion, In the very simple models under review, complex social dynamics depend significantly on the structure of the related network.. A dedicated framework such as Moduleco provides friendly environment while allowing easy simulation of various network structures, in order to explores such phenomenon in a "virtual lab".

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